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Geometric invariants of fanning curves

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ABSTRACT

We study the geometry of an important class of generic curves in the Grassmann manifolds of n-dimensional subspaces and Lagrangian subspaces of $\mathbb{R}^{2n}$ under the action of the linear and linear symplectic groups.

1. Introduction

Curves on Grassmann manifolds often appear in geometry and dynamics through the following construction. Let $\pi : E \to M$ be a fiber bundle over a manifold M and let $\phi : \mathbb{R} \times E \to E$ be a flow. If $VE \subset TE$ denotes the vertical subbundle (i.e., the kernel of $D\pi$) and e is a point in E, $t \mapsto D\phi_{-t}(V_{\phi_t(e)}E)$ is a curve of subspaces of T_eE. For example, in Riemannian and Finsler geometry $E = \mathcal{T}M$ is the punctured tangent bundle (i.e., without the zero section) of a manifold M and ϕ is the geodesic flow.

In this paper we introduce a comprehensive approach to the geometry of the class of curves in the Lagrangian Grassmannian Λ_n and the Grassmannian G_n of n-dimensional subspaces of $\mathbb{R}^{2n}$ that arise in the study of semi-sprays and Lagrangian flows. This presentation also uncovers the geome-

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try behind the formalisms of J. Klein and A. Voutier [14], J. Grifone [13], and P. Foulon [11] for the construction of connections and curvature for Finsler metrics, Lagrangian systems, and semi-sprays.

The relationship between geodesic flows, Sturm systems, and the projective differential geometry of curves in the Grassmannians G_n and Λ_n is classical and has been developed—for the most part implicitly—from different viewpoints. The explicit development of the subject seems to have been started by S. Ahdout [8] who studied it in the context of geodesic flows of Riemannian metrics and billiard maps. Later it was independently studied by V. Ovsienko [15–17] in the context of Sturm systems, and by A. Agrachev together with R. Gamkrelidze, N. Chtcherbakova, and I. Zelenko in the context of control theory (see [1–7]). Our approach encompasses, unifies, and simplifies those of our predecessors. At the same time it departs from them in that we not only study the projective geometry of curves on G_n and Λ_n, we study the genesis and geometry of the invariants themselves.

Using the canonical identification between the tangent space of the Grassmannian of n-dimensional subspaces of $\mathbb{R}^{2n}$ at an n-dimensional subspace ℓ and the space of linear maps from ℓ to the quotient space $\mathbb{R}^{2n}/\ell$, the class of curves we are interested in can be defined as follows.

Definition 1.1. A smooth curve $\ell(t)$ of n-dimensional subspaces of $\mathbb{R}^{2n}$ is said to be *fanning* if at each time t the tangent vector $\dot{\ell}(t)$ is an invertible linear map from $\ell(t)$ to the quotient space $\mathbb{R}^{2n}/\ell(t)$.

A more explicit description of these curves can be obtained by working with *frames*: if $\mathcal{A}(t)$ is a smooth curve of $2n \times n$ matrices of rank n, the curve of n-dimensional subspaces spanned by the columns of $\mathcal{A}(t)$ is fanning if and only if the $2n \times 2n$ matrix $(\mathcal{A}(t)|\dot{\mathcal{A}}(t))$—formed by juxtaposing $\mathcal{A}(t)$ and its derivative $\dot{\mathcal{A}}(t)$—is invertible for all values of t. It will be useful to denote such curves of $2n \times n$ matrices as *fanning curves of frames*, or as *fanning frames*. Remark that two fanning frames $\mathcal{A}(t)$ and $\mathcal{B}(t)$ span the same curve of n-dimensional subspaces if and only if there is a curve $X(t)$ of invertible $n \times n$ matrices such that $\mathcal{B}(t) = \mathcal{A}(t)X(t)$.

Examples.

- If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two frames such that the matrix $(\mathcal{A}_1|\mathcal{A}_2)$ is invertible, the *line* $\mathcal{A}(t) = \mathcal{A}_1 + t\mathcal{A}_2$ is a fanning frame.
- If ℓ is an n-dimensional subspace of $\mathbb{R}^{2n}$ and $\mathbf{X}$ is a linear transformation from $\mathbb{R}^{2n}$ to itself such that $\mathbf{X}\ell$ is transversal to ℓ, the curve $\ell(t) = \exp(t\mathbf{X})\ell$ is a fanning curve.
- P. Griffiths shows in [12] that the ruled surface in real projective space defined by a curve in G_2 is developable if and only if the curve in G_2 is *not* fanning on any interval.

Our main insight is that the key to understanding the geometry of fanning curves is the following, almost tautological, construction. Given a fanning curve $\ell(t)$, let π_t be the canonical projection from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}/\ell(t)$, and recall that $\dot{\ell}(t)$ is an invertible linear map from $\ell(t)$ to $\mathbb{R}^{2n}/\ell(t)$. The *fundamental endomorphism* $\mathbf{F}(t)$ associated with the fanning curve ℓ at time t is the linear transformation from $\mathbb{R}^{2n}$ to itself whose value at a vector $\mathbf{v}$ is $(\dot{\ell}(t))^{-1}\pi_t\mathbf{v} \in \ell(t) \subset \mathbb{R}^{2n}$. If the curve $\ell(t)$ is spanned by a fanning frame $\mathcal{A}(t)$, then the matrix for its fundamental endomorphism in the canonical basis of $\mathbb{R}^{2n}$ is

$$\bigl(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)\bigr)\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\bigl(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)\bigr)^{-1},$$

where 0 represents, in this case, the $n \times n$ zero matrix. It follows immediately from this formula that if $\mathbf{T}$ is an invertible linear transformation of $\mathbb{R}^{2n}$ and $\mathbf{F}(t)$ is the fundamental endomorphism of a fanning curve $\ell(t)$, the fundamental endomorphism of $\mathbf{T}\ell(t)$ is $\mathbf{T}\mathbf{F}(t)\mathbf{T}^{-1}$.

Seen as a map that takes one-jets of fanning curves to elements of the Lie algebra $\mathcal{G}l(2n)$ of $2n \times 2n$ matrices, the fundamental endomorphism describes the prolonged action of $GL(2n)$ on the space of one-jets of fanning curves on the Grassmannian. Indeed, in Section 7 we shall prove the following characterization.

Theorem 1.2. *A map from the space of one-jets of fanning curves in* G_n *to the Lie algebra* $\mathcal{G}l(2n)$ *is equivariant with respect to the* $\mathrm{GL}(2n)$ *action on these spaces if and only if it is of the form* $a\mathbf{F} + b\mathbf{I}$*, where* $\mathbf{I}$ *is the identity matrix and* a *and* b *are real numbers.*

The curve $\mathbf{F}(t)$ associated to a fanning curve $\ell(t)$ has the remarkable property that its derivative $\dot{\mathbf{F}}(t)$ is a curve of reflections (i.e., $(\dot{\mathbf{F}}(t))^2 = \mathbf{I}$) such that the eigenspace of $\dot{\mathbf{F}}(t)$ associated to the eigenvalue -1 is precisely $\ell(t)$. The eigenspace of $\dot{\mathbf{F}}(t)$ associated to the eigenvalue 1 is the *horizontal curve* $h(t)$ of $\ell(t)$.

The horizontal curve—appearing for the first time, under a different guise, in the work of S. Ahdout [8]—plays a fundamental role in the understanding of the geometric invariants of $\ell(t)$. In fact, we will see in Section 7 that it describes the prolonged action of $\mathrm{GL}(2n)$ on the space of two-jets of fanning curves on the Grassmannian.

Theorem 1.3. *The assignment that sends a fanning curve* $\ell(t)$ *to its horizontal curve* $h(t)$ *is characterized by the following four properties*:

(1) *At each time* t *the subspace* $h(t)$ *is transversal to* $\ell(t)$*.*
(2) *The subspace* $h(\tau)$ *depends only on the two-jet of the curve* $\ell(t)$ *at* $t = \tau$*.*
(3) *If* $\mathbf{T}$ *is an invertible linear transformation of* $\mathbb{R}^{2n}$*, the horizontal curve of* $\mathbf{T}\ell(t)$ *is* $\mathbf{T}h(t)$*.*
(4) *If* $\ell(t)$ *is spanned by a line* $\mathcal{A} + t\mathcal{B}$ *in the space of frames,* $h(t)$ *is constant.*

The main geometric invariant of a fanning curve $\ell(t)$ in the Grassmannian is its *Jacobi endomorphism* $\mathbf{K}(t) = \ddot{\mathbf{F}}(t)^2/4$. Alternatively, if $\mathbf{P}(t) = (\mathbf{I} - \dot{\mathbf{F}}(t))/2$ is the projection onto $\ell(t)$ with kernel $h(t)$, $\mathbf{K}(t) = \dot{\mathbf{P}}(t)^2$. Hence, the Jacobi endomorphism describes how the horizontal curve moves with respect to $\ell(t)$.

Theorem 1.4. *Let* $\ell(t)$ *be a fanning curve in the Grassmannian* G_n *and let* $h(t)$ *be its horizontal curve. The Jacobi endomorphism* $\mathbf{K}(t)$ *of* $\ell(t)$ *satisfies the following properties*:

(1) *At each time* t *the endomorphism* $\mathbf{K}(t)$ *preserves the decomposition* $\mathbb{R}^{2n} = \ell(t) \oplus h(t)$*.*
(2) *If* $\mathbf{T}$ *is an invertible linear map from* $\mathbb{R}^{2n}$ *to itself, the Jacobi endomorphism of* $\mathbf{T}\ell(t)$ *is* $\mathbf{T}\mathbf{K}(t)\mathbf{T}^{-1}$*.*
(3) *If* s *is a diffeomorphism of the real line and* $\{s(t), t\}$ *denotes its Schwarzian derivative, the Jacobi endomorphism of* $\ell(s(t))$ *is*

$$\mathbf{K}\big(s(t)\big)\dot{s}(t)^2 + (1/2)\big\{s(t), t\big\}\mathbf{I}.$$

The Jacobi endomorphism clarifies the geometry underlying the Schwarzian derivative and its matrix generalizations. If $\mathcal{A}(t)$ is a fanning frame, at each instant t the columns of $\mathcal{A}(t)$ and $\dot{\mathcal{A}}(t)$ span $\mathbb{R}^{2n}$ and, therefore, we have the differential equation

$$\ddot{\mathcal{A}} + \dot{\mathcal{A}}P(t) + \mathcal{A}Q(t) = 0, \tag{1.1}$$

where $Q(t)$ and $P(t)$ are smooth curves of $n \times n$ matrices. Let us define the *Schwarzian* of $\mathcal{A}(t)$ as the function

$$\big\{\mathcal{A}(t), t\big\} = 2Q(t) - (1/2)P(t)^2 - \dot{P}(t). \tag{1.2}$$

When the fanning frame is of the form $\mathcal{A}(t) = \binom{I}{M(t)}$, then

$$\big\{\mathcal{A}(t), t\big\} = \frac{d}{dt}\big(\dot{M}^{-1}\ddot{M}\big) - (1/2)\big(\dot{M}^{-1}\ddot{M}\big)^2$$

is the matrix Schwarzian introduced by B. Schwarz [18] and M.I. Zelikin [20].

Theorem 1.5. *The Schwarzian of a fanning frame $\mathcal{A}(t)$ is characterized by the equation $\mathbf{K}(t)\mathcal{A}(t) = (1/2)\mathcal{A}(t)\{\mathcal{A}(t), t\}$.*

The properties of the Schwarzian follow immediately from those of the Jacobi endomorphism.

Corollary 1.6. *The Schwarzian of a fanning frame $\mathcal{A}(t)$ satisfies the following properties*:

(1) *If* $\mathbf{T}$ *is an invertible linear transformation from* $\mathbb{R}^{2n}$ *to itself,* $\{\mathbf{T}\mathcal{A}(t), t\} = \{\mathcal{A}(t), t\}$.
(2) *If* $X(t)$ *is a curve of invertible* $n \times n$ *matrices,* $\{\mathcal{A}(t)X(t), t\} = X(t)^{-1}\{\mathcal{A}(t), t\}X(t)$.
(3) *If* s *is a diffeomorphism of the real line,* $\{\mathcal{A}(s(t)), t\} = \{A(s(t)), s\}\dot{s}(t)^2 + \{s(t), t\}I$.

Because of property (2), the only thing we can say about fanning frames spanning congruent curves in G_n is that their Schwarzians are pointwise conjugate. In order to obtain a deeper understanding of the geometry of fanning curves, in Section 4 we introduce *normal frames* for which Eq. (1.1) takes the normal form

$$\ddot{\mathcal{A}} + (1/2)\mathcal{A}\big\{\mathcal{A}(t), t\big\} = 0. \tag{1.3}$$

Using normal frames, the "fundamental theorem" for fanning curves on the Grassmannian is an easy consequence of the uniqueness theorem for solutions of differential equations.

Theorem 1.7. *Two fanning curves of* n*-dimensional subspaces of* $\mathbb{R}^{2n}$ *are congruent if and only if the Schwarzians of any two of their normal frames are conjugate by a constant* $n \times n$ *invertible matrix.*

The fundamental theorem for curves of Lagrangian subspaces under the action of the linear symplectic group is only slightly more involved and will be presented in Section 6.

The Jacobi endomorphism is also central to the study of unparameterized fanning curves in G_n. Again, this is a classical subject—when $n = 2$ this is the projective differential geometry of nondevelopable ruled surfaces in $\mathbb{R}P^3$ (see, for example, [19])—but the new approach eliminates many of the computations and gives a better understanding of the subject.

It is clear from property (3) in Theorem 1.4 that any fanning curve admits a *special parameterization* so that the trace of its Jacobi endomorphism vanishes identically. Moreover, any two such parameterizations are projectively equivalent. It follows that in a special parameterization the operator-valued quadratic differential $\mathbf{K}(t)\,dt^2$ is an invariant of the unparameterized curve. Defining a *special normal frame* as a normal frame that spans a fanning curve with a special parameterization, the fundamental theorem for unparameterized fanning curves may be stated as follows.

Theorem 1.8. *Two unparameterized fanning curves of* n*-dimensional subspaces of* $\mathbb{R}^{2n}$ *are congruent if and only if up to a projective change of parameters the Schwarzians of any two of their special normal frames are conjugate by a constant* $n \times n$ *invertible matrix.*

In the case $n = 2$ this result is due to Wilczynski (see [19, pp. 114–116]). Indeed, in some sense the fundamental theorems for fanning curves in G_n and Λ_n, parameterized or not, are just geometric reinterpretations of old work on the invariant theory of linear ordinary differential equations. We have included them in this paper because after a long search of the literature we were unable to find suitable references.

Also in this paper the reader will find a characterization of fanning curves in G_n that are projections of one-parameter subgroups of the linear group $\mathrm{GL}(2n)$ (Section 5), a thorough study of the geometry of fanning curves of Lagrangian subspaces (Section 6), and a simplified account of the approaches of S. Ahdout and A.A. Agrachev et al. to the geometry of fanning curves (Section 8). In our opinion, the additional insight that a comparisons of approaches will give the reader warrants this small scholarly effort on our part. We also think this will help break the cycle of rediscovery in which the subject has been caught up for some time.

The close relationship between the geometry of fanning curves in the Grassmannian and the elegant formalisms of connections for second-order differential equations (semi-sprays) developed by J. Klein and A. Voutier [14], J. Grifone [13], and P. Foulon [11] will be the subject of a future publication [9]. At the end of this lengthy introduction we can succinctly summarize the contents of this paper and of that future publication as follows:

> the geometric invariants of fanning curves in the Grassmannian on n-dimensional subspaces of $\mathbb{R}^{2n}$ arise from the fundamental endomorphism $\mathbf{F}$—a tautological construct—and its derivatives $\dot{\mathbf{F}}$ and $\ddot{\mathbf{F}}$, from which we define the Jacobi endomorphism $\mathbf{K} = (1/4)\ddot{\mathbf{F}}^2$. Using the construction outlined in the first paragraph of the introduction, these correspond in a precise way to the vertical endomorphism, connection, and curvature (Jacobi endomorphism) introduced by Klein, Voutier, Grifone and Foulon in their study of connections for semi-sprays and Finsler metrics.

2. The fundamental endomorphism and its derivatives

The two groups acting naturally on the space of fanning curves in G_n are the group of invertible linear transformations of $\mathbb{R}^{2n}$ and the group of diffeomorphisms of the real line acting by reparameterizations. If we wish to work with fanning frames, we must add the group of smooth curves of $n \times n$ invertible matrices acting by $(\mathcal{A}(t), X(t)) \mapsto \mathcal{A}(t)X(t)$. Any quantity associated to a fanning frame $\mathcal{A}(t)$ that is invariant under this action depends only on the fanning curve on the Grassmannian defined by the span of the columns of $\mathcal{A}(t)$.

We caution the reader that unless we specifically state otherwise the canonical basis of $\mathbb{R}^{2n}$ shall be freely used to identify linear transformations of $\mathbb{R}^{2n}$ with $2n \times 2n$ matrices. Despite this, all constructions are intrinsic in nature and apply to curves of n-dimensional linear subspaces in a $2n$-dimensional vector space over the real numbers.

Definition 2.1. The *fundamental endomorphism* of a fanning frame $\mathcal{A}(t)$ at a given time τ is the linear transformation from $\mathbb{R}^{2n}$ to itself defined by the equations $\mathbf{F}(\tau)\mathcal{A}(\tau) = 0$, $\mathbf{F}(\tau)\dot{\mathcal{A}}(\tau) = \mathcal{A}(\tau)$.

Equivalently, as in the introduction, we could have defined the fundamental endomorphism by the formula

$$\mathbf{F}(t) = \big(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)\big)\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\big(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)\big)^{-1}.$$

Using this formula, an easy calculation suffices to establish the main properties of the fundamental endomorphism.

Proposition 2.2. *Let $\mathcal{A}(t)$ be a fanning frame. Its fundamental endomorphism $\mathbf{F}(t)$ satisfies the following properties*:

(1) *If $X(t)$ is a smooth curve of invertible $n \times n$ matrices, the fundamental endomorphism of $\mathcal{A}(t)X(t)$ is again $\mathbf{F}(t)$.*
(2) *If $\mathbf{T}$ is an invertible linear transformation from $\mathbb{R}^{2n}$ to itself, the fundamental endomorphism of $\mathbf{T}\mathcal{A}(t)$ is $\mathbf{T}\mathbf{F}(t)\mathbf{T}^{-1}$.*
(3) *The fundamental endomorphism of the reparameterized curve $\mathcal{A}(s(t))$ is $\mathbf{F}(s(t))\dot{s}(t)^{-1}$.*

By the first property, the fundamental endomorphism is defined for fanning curves in the Grassmannian. The intrinsic definition given in Section 1—suggested to us by Fran Burstall—justifies the term "fundamental." On the other hand, basic properties such as the one given in the following proposition are hard to prove intrinsically.

Proposition 2.3. *Let $\mathbf{F}(t)$ be the fundamental endomorphism of a fanning frame $\mathcal{A}(t)$. At each value of t the derivative $\dot{\mathbf{F}}(t)$ is a reflection* (*i.e.*, $(\dot{\mathbf{F}}(t))^2 = \mathbf{I}$)) *whose -1 eigenspace is spanned by the columns of $\mathcal{A}(t)$.*

Proof. We first show that $\dot{\mathbf{F}}(t)\mathcal{A}(t) = -\mathcal{A}(t)$. From this follows that $\dot{\mathbf{F}}(t)$ restricted to the subspace $\ell(t)$ spanned by the columns of $\mathcal{A}(t)$ is minus the identity. Since $\ell(t)$ is the range of $\mathbf{F}(t)$, this also implies that $\dot{\mathbf{F}}(t)\mathbf{F}(t) = -\mathbf{F}(t)$. Differentiating the equation $\mathbf{F}(t)\mathcal{A}(t) = 0$ and recalling that $\mathbf{F}(t)\dot{\mathcal{A}}(t) = \mathcal{A}(t)$, it follows that $\dot{\mathbf{F}}(t)\mathcal{A}(t) + \mathbf{F}(t)\dot{\mathcal{A}}(t) = 0$ and $\dot{\mathbf{F}}(t)\mathcal{A}(t) = -\mathcal{A}(t)$.

We now show that $(\dot{\mathbf{F}}(t))^2\dot{\mathcal{A}}(t) = \dot{\mathcal{A}}(t)$. Since we already know that $(\dot{\mathbf{F}}(t))^2\mathcal{A}(t) = \mathcal{A}(t)$, this will prove that $\dot{\mathbf{F}}(t)$ is a reflection for every value of t. Differentiating the equation $\mathbf{F}(t)\dot{\mathcal{A}}(t) = \mathcal{A}(t)$ and using that $\dot{\mathbf{F}}(t)\mathbf{F}(t) = -\mathbf{F}(t)$, we have that

$$\big(\dot{\mathbf{F}}(t)\big)^2\dot{\mathcal{A}}(t) = \dot{\mathbf{F}}(t)\big(\dot{A}(t) - \mathbf{F}(t)\ddot{\mathcal{A}}(t)\big) = \dot{A}(t). \qquad \square$$

We remark that since the fundamental endomorphism depends only on the curve on the Grassmannian and not on the fanning frame used to represent it, the same holds for all its time derivatives. In particular, the curve of reflections $\dot{\mathbf{F}}(t)$ and the associated curve of projections $\mathbf{P}(t) = (\mathbf{I} - \dot{\mathbf{F}}(t))/2$ depend only on the curve on the Grassmannian.

Definition 2.4. Let $\ell(t)$ be a fanning curve in the Grassmannian and let $\mathbf{F}(t)$ be its fundamental endomorphism. The map that takes t to the kernel of the projection $\mathbf{P}(t) = (\mathbf{I} - \dot{\mathbf{F}}(t))/2$ is the *horizontal curve* of $\ell(t)$.

It is clear from the definition that the subspace $h(\tau)$ is transversal to $\ell(\tau)$ and depends only on the two-jet of $\ell(t)$ at $t = \tau$.

Proposition 2.5. *Let $\ell(t)$ be a fanning curve on G_n and let $h(t)$ be its horizontal curve. If $\mathbf{T}$ is an invertible linear transformation from $\mathbb{R}^{2n}$ to itself, the horizontal curve of $\mathbf{T}\ell(t)$ is $\mathbf{T}h(t)$.*

Proof. This follows immediately from the fact that if $\mathbf{T}$ is an invertible linear transformation from $\mathbb{R}^{2n}$ to itself, the fundamental endomorphism of $\mathbf{T}\ell(t)$ and its derivative are $\mathbf{T}\mathbf{F}(t)\mathbf{T}^{-1}$ and $\mathbf{T}\dot{\mathbf{F}}(t)\mathbf{T}^{-1}$, respectively. $\square$

We now turn to the study of the second derivative $\ddot{\mathbf{F}}$ of the fundamental endomorphism. The geometric meaning of the computations will be clearer if we work with $\dot{\mathbf{P}} = -(1/2)\ddot{\mathbf{F}}$.

Proposition 2.6. *Let $\ell(t)$ be a fanning curve in the Grassmannian G_n and let $h(t)$ be its horizontal curve. If $\mathbf{P}(t)$ denotes the projection onto $\ell(t)$ with kernel $h(t)$, then $\dot{\mathbf{P}}(t)$ maps $h(t)$ into $\ell(t)$ and maps $\ell(t)$ isomorphically to $h(t)$.*

Proof. Differentiating the identity $\mathbf{P}(t)^2 = \mathbf{P}(t)$ we have that $\dot{\mathbf{P}}(t)\mathbf{P}(t) = (\mathbf{I} - \mathbf{P}(t))\dot{\mathbf{P}}(t)$. If we notice that $\mathbf{I} - \mathbf{P}(t)$ is the projection onto $h(t)$ with kernel $\ell(t)$, the equation

$$\dot{\mathbf{P}}(t)\big(\ell(t)\big) = \dot{\mathbf{P}}(t)\mathbf{P}(t)\big(\ell(t)\big) = \big(\mathbf{I} - \mathbf{P}(t)\big)\dot{\mathbf{P}}(t)\big(\ell(t)\big)$$

implies that the subspace $\dot{\mathbf{P}}(t)(\ell(t))$ is contained in $h(t)$. The proof that the subspace $\dot{\mathbf{P}}(t)(h(t))$ is contained in $\ell(t)$ is nearly identical.

In order to verify that $\dot{\mathbf{P}}(t)$ maps $\ell(t)$ isomorphically onto $h(t)$, we make use of the identity $\dot{\mathbf{P}} = (-1/2)\ddot{\mathbf{F}}$. It follows from the proof of Proposition 2.3 that if $\mathcal{A}(t)$ is a fanning frame spanning $\ell(t)$, then $\dot{\mathbf{F}}(t)\mathcal{A}(t) = -\mathcal{A}(t)$ and $\dot{\mathbf{F}}(t)\dot{\mathcal{A}}(t) = \dot{\mathcal{A}}(t) - \mathbf{F}(t)\ddot{\mathcal{A}}(t)$. Differentiating the first of these equations and using the second, we obtain $\dot{\mathbf{P}}(t)\mathcal{A}(t) = \dot{\mathcal{A}}(t) - (1/2)\mathbf{F}(t)\ddot{\mathcal{A}}(t)$. Since the columns of $\mathbf{F}(t)\ddot{\mathcal{A}}(t)$ are linear combinations of those of $\mathcal{A}(t)$, and $(\mathcal{A}(t)|\dot{\mathcal{A}}(t))$ is invertible, it follows that $\dot{\mathbf{P}}(t)\mathcal{A}(t)$ has rank n and must span $h(t)$. $\square$

A remark on the preceding proof is that $\dot{\mathbf{P}}(t)\mathcal{A}(t)$ is the projection of $\dot{\mathcal{A}}(t)$ onto the horizontal subspace $h(t)$. Indeed, on differentiating the equality $\mathbf{P}(t)\mathcal{A}(t) = \mathcal{A}(t)$, we obtain $\dot{\mathbf{P}}(t)\mathcal{A}(t) = (\mathbf{I} - \mathbf{P}(t))\dot{\mathcal{A}}(t)$.

Definition 2.7. The *horizontal derivative* of a fanning frame $\mathcal{A}(t)$ is the curve of frames

$$\mathcal{H}(t) := \big(\mathbf{I} - \mathbf{P}(t)\big)\dot{\mathcal{A}}(t) = \dot{\mathbf{P}}(t)\mathcal{A}(t) = \dot{\mathcal{A}}(t) - (1/2)\mathbf{F}(t)\ddot{\mathcal{A}}(t) = -(1/2)\ddot{\mathbf{F}}(t)\mathcal{A}(t).$$

Remark. If, as in Eq. (1.1), we write $\ddot{\mathcal{A}} + \dot{\mathcal{A}}P(t) + \mathcal{A}Q(t) = 0$, then the horizontal derivative of $\mathcal{A}(t)$ is

$$\mathcal{H}(t) = \dot{\mathcal{A}}(t) + (1/2)\mathcal{A}(t)P(t). \tag{2.1}$$

From the horizontal derivative we immediately obtain an elementary description of the horizontal subspace of a fanning curve.

Proposition 2.8. *Let $\ell(t)$ be a fanning curve on the Grassmannian. If $\mathcal{A}_\tau(t)$ is a fanning frame spanning $\ell(t)$ that satisfies $\ddot{\mathcal{A}}_\tau(\tau) = 0$, then the columns of $\dot{\mathcal{A}}_\tau(\tau)$ span the horizontal subspace of $\ell(t)$ at $t = \tau$.*

In the two-dimensional case this implies that if $\ell(t)$ is a fanning curve in the projective line and $h(t)$ is its horizontal curve, then a straight line l_τ on the plane not passing through the origin is parallel to $h(\tau)$ if and only if the acceleration of the curve of vectors $\mathbf{v}_\tau(t) = \ell(t) \cap l_\tau$ is zero at $t = \tau$.

An easy calculation shows that the horizontal derivative is well-behaved with respect to the three natural group actions on the space of fanning frames.

Proposition 2.9. *The horizontal derivative $\mathcal{H}(t)$ of a fanning frame $\mathcal{A}(t)$ satisfies the following properties*:

(1) *If $X(t)$ is a smooth curve of $n \times n$ invertible matrices, the horizontal derivative of $\mathcal{A}(t)X(t)$ is $\mathcal{H}(t)X(t)$.*
(2) *If $\mathbf{T}$ is an invertible linear transformation from $\mathbb{R}^{2n}$ to itself, the horizontal derivative of $\mathbf{T}\mathcal{A}(t)$ is $\mathbf{T}\mathcal{H}(t)$.*
(3) *The horizontal derivative of the reparameterized curve $\mathcal{A}(s(t))$ is*

$$\mathcal{H}\big(s(t)\big)\dot{s}(t) + (1/2)\mathcal{A}\big(s(t)\big)\dot{s}(t)^{-1}\ddot{s}(t).$$

The horizontal derivative is useful in many computations like the one in the proof of the following interesting property of the fundamental endomorphism and its derivatives.

Proposition 2.10. *Let $\ell(t)$ be a fanning curve with fundamental endomorphism $\mathbf{F}(t)$. If $[\mathbf{A}, \mathbf{B}]_+$ denotes the anticommutator $\mathbf{AB} + \mathbf{BA}$, then $[\mathbf{F}(t), \dot{\mathbf{F}}(t)]_+ = 0$, $[\dot{\mathbf{F}}(t), \ddot{\mathbf{F}}(t)]_+ = 0$, and $[\mathbf{F}(t), \ddot{\mathbf{F}}(t)]_+ = -2\mathbf{I}$.*

Proof. The first two identities are obtained by differentiating the identities $\mathbf{F}(t)^2 = 0$ and $\dot{\mathbf{F}}(t)^2 = \mathbf{I}$. To obtain the third we take a curve of frames $\mathcal{A}(t)$ spanning $\ell(t)$ and compute both $[\mathbf{F}(t), \ddot{\mathbf{F}}(t)]_+\mathcal{A}(t)$ and $[\mathbf{F}(t), \ddot{\mathbf{F}}(t)]_+\mathcal{H}(t)$. In the first of these anticommutators, notice that $\ddot{\mathbf{F}}\mathbf{F}\mathcal{A} = 0$ and that

$$\mathbf{F}\ddot{\mathbf{F}}\mathcal{A} = \mathbf{F}(-2\mathcal{H}) = \mathbf{F}(-2\dot{\mathcal{A}} + \mathbf{F}\ddot{\mathcal{A}}) = -2\mathcal{A}.$$

In the second anticommutator, we have that $\mathbf{F}\ddot{\mathbf{F}}\mathcal{H} = -2\mathbf{F}\dot{\mathbf{P}}\mathcal{H} = 0$ because $\dot{\mathbf{P}}$ sends $h(t)$ into $\ell(t)$, the kernel of $\mathbf{F}(t)$. Moreover,

$$\ddot{\mathbf{F}}\mathbf{F}\mathcal{H} = \ddot{\mathbf{F}}\mathbf{F}\big(\dot{\mathcal{A}} - (1/2)\mathbf{F}\ddot{\mathcal{A}}\big) = \ddot{\mathbf{F}}\mathcal{A} = -2\mathcal{H}.$$

We have then that $[\dot{\mathbf{F}}(t), \ddot{\mathbf{F}}(t)]_+\mathcal{A}(t) = -2\mathcal{A}(t)$ and that $[\dot{\mathbf{F}}(t), \ddot{\mathbf{F}}(t)]_+\mathcal{H}(t) = -2\mathcal{H}(t)$, which means that $[\dot{\mathbf{F}}(t), \ddot{\mathbf{F}}(t)]_+ = -2\mathbf{I}$. $\square$

3. The Jacobi endomorphism and the Schwarzian

The natural notion of curvature for fanning curves in the Grassmannian G_n is given by an operator-valued function of the parameter that is closely related to the Schwarzian derivative and its matrix generalizations.

Definition 3.1. Let $\ell(t)$ be a fanning curve in the Grassmannian G_n, let $\mathbf{F}(t)$ be its fundamental endomorphism, and let $h(t)$ be its horizontal curve. The *Jacobi endomorphism* of $\ell(t)$ is $\mathbf{K}(t) = \ddot{\mathbf{F}}(t)^2/4$. Alternatively, if $\mathbf{P}(t) = (\mathbf{I} - \dot{\mathbf{F}}(t))/2$ is the projection onto $\ell(t)$ with kernel $h(t)$, $\mathbf{K}(t) = \dot{\mathbf{P}}(t)^2$.

The intuition behind the Jacobi endomorphism is that it measures how the horizontal curve moves with respect to $\ell(t)$. More precisely, if $\mathcal{A}(t)$ is a fanning frame spanning $\ell(t)$ and $\mathcal{H}(t)$ is its horizontal derivative,

$$\mathbf{P}(t)\dot{\mathcal{H}}(t) = -\dot{\mathbf{P}}(t)\mathcal{H}(t) = -\dot{\mathbf{P}}(t)^2\mathcal{A}(t) = -\mathbf{K}(t)\mathcal{A}(t). \tag{3.1}$$

Theorem 3.2. *Let $\ell(t)$ be a fanning curve in G_n and let $h(t)$ be its horizontal curve. The Jacobi endomorphism $\mathbf{K}(t)$ of $\ell(t)$ satisfies the following properties*:

(1) *At each time t the endomorphism $\mathbf{K}(t)$ preserves the decomposition $\mathbb{R}^{2n} = \ell(t) \oplus h(t)$.*
(2) *If $\mathbf{T}$ is an invertible linear map from $\mathbb{R}^{2n}$ to itself, the Jacobi endomorphism of $\mathbf{T}\ell(t)$ is $\mathbf{T}\mathbf{K}(t)\mathbf{T}^{-1}$.*
(3) *If s is a diffeomorphism of the real line, the Jacobi endomorphism of $\ell(s(t))$ is*

$$\mathbf{K}\big(s(t)\big)\dot{s}(t)^2 + (1/2)\big\{s(t), t\big\}\mathbf{I}.$$

Proof. The first property follows immediately from the identity $\mathbf{K}(t) = \dot{\mathbf{P}}(t)^2$ and Proposition 2.6. The second property follows from the $GL(2n)$-equivariance of the fundamental endomorphism.

By Proposition 2.2, if $\mathbf{F}(t)$ denotes the fundamental endomorphism of $\ell(t)$, the fundamental endomorphism of $\ell(s(t))$ is $\mathbf{F}(s(t))\dot{s}(t)^{-1}$. Setting, for convenience, $r(t) = \dot{s}(t)^{-1}$, we have that the second derivative of the fundamental endomorphism of $\ell(s(t))$ is

$$\mathbf{F}''\big(s(t)\big)\dot{s}(t) + \mathbf{F}'(t)\dot{s}(t)\dot{r}(t) + \mathbf{F}\big(s(t)\big)\ddot{r}(t), \quad \text{where } ' = d/ds.$$

Writing the square of this expression in terms of anticommutators and applying the identities of Proposition 2.10 together with the identities $\mathbf{F}^2 = 0$ and $(\mathbf{F}')^2 = \mathbf{I}$, we obtain

$$\left(\frac{d^2}{dt^2}\mathbf{F}\big(s(t)\big)\dot{s}(t)^{-1}\right)^2 = \mathbf{F}''\big(s(t)\big)^2\dot{s}(t)^2 + \mathbf{I}\big(\dot{s}(t)^2\dot{r}(t)^2 - 2\dot{s}(t)\ddot{r}(t)\big).$$

A short calculation reveals that $(\dot{s}(t)^2\dot{r}(t)^2 - 2\dot{s}(t)\ddot{r}(t))$ is twice the Schwarzian $\{s(t), t\}$. We conclude that the Jacobi endomorphism of $\ell(s(t))$ is $\mathbf{K}(s(t))\dot{s}(t)^2 + (1/2)\{s(t), t\}\mathbf{I}$. $\square$

Definition 3.3. If $\mathcal{A}(t)$ is a fanning frame and the curves of $n \times n$ matrices $Q(t)$ and $P(t)$ are defined by the equation $\ddot{\mathcal{A}} + \dot{\mathcal{A}}P(t) + \mathcal{A}Q(t) = 0$, the function

$$\big\{\mathcal{A}(t), t\big\} = 2Q(t) - (1/2)P(t)^2 - \dot{P}(t)$$

is the *Schwarzian* of $\mathcal{A}(t)$.

Theorem 3.4. *The Schwarzian of a fanning frame $\mathcal{A}(t)$ is characterized by the equation $\mathbf{K}(t)\mathcal{A}(t) = (1/2)\mathcal{A}(t)\{\mathcal{A}(t), t\}$.*

Proof. The proof follows from the equations $\mathbf{K}(t)\mathcal{A}(t) = -\mathbf{P}(t)\dot{\mathcal{H}}(t)$ and $\mathcal{H}(t) = \dot{\mathcal{A}}(t) + (1/2)\mathcal{A}(t)P(t)$. Indeed, on differentiating this last equation and replacing $\ddot{\mathcal{A}}(t)$ by $-\dot{\mathcal{A}}(t)P(t) - \mathcal{A}(t)Q(t)$, we obtain

$$\dot{\mathcal{H}}(t) = -(1/2)\mathcal{A}(t)\{\mathcal{A}(t), t\} - (1/2)\mathcal{H}(t)P(t). \tag{3.2}$$

Applying $-\mathbf{P}(t)$ to both sides of the equation yields the result. □

Corollary 3.5. *If $\mathcal{A}(t)$ is a fanning frame and $\mathcal{H}(t)$ is its horizontal derivative, the matrix for the Jacobi endomorphism $\mathbf{K}(t)$ of $\mathcal{A}(t)$ in the basis of $\mathbb{R}^{2n}$ formed by the columns of $(\mathcal{A}(t)|\mathcal{H}(t))$ is*

$$\begin{pmatrix} \{\mathcal{A}(t), t\}/2 & 0 \\ 0 & \{\mathcal{A}(t), t\}/2 \end{pmatrix}.$$

Proof. Since we already know that $\mathbf{K}(t)\mathcal{A}(t) = (1/2)\mathcal{A}(t)\{\mathcal{A}(t), t\}$, it remains to compute $\mathbf{K}(t)\mathcal{H}(t) = \dot{\mathbf{P}}(t)^2\mathcal{H}(t)$. By Eq. (3.1), $\dot{\mathbf{P}}\mathcal{H}(t) = \mathbf{K}(t)\mathcal{A}(t) = (1/2)\mathcal{A}(t)\{\mathcal{A}(t), t\}$. This yields

$$\mathbf{K}(t)\mathcal{H}(t) = \dot{\mathbf{P}}(t)^2\mathcal{H}(t) = (1/2)\dot{\mathbf{P}}(t)\mathcal{A}(t)\{\mathcal{A}(t), t\} = (1/2)\mathcal{H}(t)\{\mathcal{A}(t), t\}. \qquad \square$$

The properties of the Schwarzian follow immediately from those of the Jacobi endomorphism.

Corollary 3.6. *The Schwarzian of a fanning frame $\mathcal{A}(t)$ satisfies the following properties*:

(1) *If $\mathbf{T}$ is an invertible linear transformation from $\mathbb{R}^{2n}$ to itself, $\{\mathbf{T}\mathcal{A}(t), t\} = \{\mathcal{A}(t), t\}$.*
(2) *If $X(t)$ is a curve of invertible $n \times n$ matrices, $\{\mathcal{A}(t)X(t), t\} = X(t)^{-1}\{\mathcal{A}(t), t\}X(t)$.*
(3) *If s is a diffeomorphism of the real line, $\{\mathcal{A}(s(t)), t\} = \{A(s(t)), s\}\dot{s}(t)^2 + \{s(t), t\}I$.*
(4) *If the fanning frame is of the form $\mathcal{A}(t) = \binom{I}{M(t)}$, then*

$$\{\mathcal{A}(t), t\} = \frac{d}{dt}(\dot{M}^{-1}\ddot{M}) - (1/2)(\dot{M}^{-1}\ddot{M})^2.$$

As a simple application of these properties, we obtain the precise—and apparently new—transformation law for the matrix Schwarzian under matrix linear fractional transformations (compare with [21, Proposition 6.14, p. 207]).

Corollary 3.7. *Let A, B, C, and D be $n \times n$ matrices, and let $M(t)$ be a smooth curve of $n \times n$ matrices such that its derivative $\dot{M}(t)$ is never singular. If $(A + BM(t))$ is invertible for all values of t in some interval and if $S_t(M)$ denotes the matrix Schwarzian of $M(t)$, then the matrix Schwarzian of $(C + DM(t))(A + BM(t))^{-1}$ on that interval is $(A + BM(t))S_t(M)(A + BM(t))^{-1}$.*

Proof. By the preceding corollary, the matrix Schwarzian of $(C + DM(t))(A + BM(t))^{-1}$ is the Schwarzian of the fanning frame

$$\mathcal{A}(t) = \begin{pmatrix} A + BM(t) \\ C + DM(t) \end{pmatrix}(A + BM(t))^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I \\ M(t) \end{pmatrix}(A + BM(t))^{-1}.$$

The result now follows from the first two properties of the Schwarzian of frames (Corollary 3.6). □

3.1. Reparameterization invariants of fanning curves

The reader has undoubtedly noticed that the traces of powers and derivatives of the Jacobi endomorphism of a fanning curve do not change if the curve is subjected to the action of linear transformations on the Grassmannian G_n. However, these quantities are not invariant under reparameterizations of the curve.

The simplest quantity associated to a fanning curve that is invariant under both reparameterizations and the action of linear transformations on the Grassmannian G_n is the fourth-order differential

$$\left(\operatorname{tr}\mathbf{K}(t)^2 - (1/n)\left(\operatorname{tr}\mathbf{K}(t)\right)^2\right)(dt)^4.$$

Under a different notation, this invariant appeared for the first time in Agrachev and Zelenko [5].

Another interesting invariant (or rather covariant) is given by the operator-valued, fifth-order differential $[\dot{\mathbf{K}}(t), \mathbf{K}(t)](dt)^5$ which is the analogue for fanning curves of a projective invariant for semi-sprays that is described by Foulon [11, Proposition VI.5].

The proof of the invariance of the fourth and fifth order differentials just defined is a simple computation involving property (3) in Theorem 3.2. However, the most important information that can be extracted from this property is that any fanning curve admits a *special parameterization.*

Definition 3.8. A fanning curve $\ell(t)$ is said to be *specially parameterized* if the trace of its Jacobi endomorphism vanishes identically.

Two special parameterizations are related by a projective transformation and, therefore, in a special parameterization the operator-valued quadratic differential $\mathbf{K}(t)\,dt^2$ is an invariant of the unparameterized curve. We shall see in the next section in what way this is the "fundamental invariant" for unparameterized fanning curves in G_n.

4. Normal frames and the congruence problem

As we have remarked all along, if a fanning curve $\ell(t)$ on the Grassmannian is spanned by a frame $\mathcal{A}(t)$ it is also spanned by all frames of the form $\mathcal{A}(t)X(t)$, where $X(t)$ is a smooth curve of invertible $n \times n$ matrices. Since $\{\mathcal{A}(t)X(t), t\} = X(t)^{-1}\{\mathcal{A}(t), t\}X(t)$, the Schwarzians of any two frames for $\ell(t)$ are point-wise conjugate. Hence, quantities such as the traces of the Schwarzian and its powers will depend only on the fanning curve on the Grassmannian and are, furthermore, invariant under the action of the linear group on the space of fanning curves. Nevertheless, if we wish to obtain a deeper understanding of the geometry of a fanning curve, we need to introduce a class of frames that is better adapted to it.

Definition 4.1. A fanning frame $\mathcal{A}(t)$ is said to be *normal* if the columns of its second derivative $\ddot{\mathcal{A}}(t)$ are linear combinations of the columns of $\mathcal{A}(t)$ for all values of t.

Proposition 4.2. *If $\ell(t)$ is a fanning curve of n-dimensional subspaces in $\mathbb{R}^{2n}$, there exists a normal frame that spans it. Moreover, if $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are two normal frames spanning $\ell(t)$, there exists a fixed invertible $n \times n$ matrix X such that $\mathcal{B}(t) = \mathcal{A}(t)X$.*

Proof. Let $\mathcal{A}(t)$ be any fanning frame spanning $\ell(t)$ and let $P(t)$ be the curve of $n \times n$ matrices defined by the equation $\ddot{\mathcal{A}}(t) + \dot{\mathcal{A}}(t)P(t) + \mathcal{A}(t)Q(t) = 0$. If $X(t)$ is the curve of $n \times n$ matrices that solves the initial value problem $\dot{X}(t) = (1/2)P(t)X(t)$, $X(0) = I$, an easy computation shows that $\mathcal{A}(t)X(t)$ is a normal frame. Notice that since $X(0)$ is invertible, $X(t)$ is invertible for all values of t and $\mathcal{A}(t)X(t)$ is again a fanning frame.

If $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are fanning frames spanning the same curve $\ell(t)$ in G_n, then there exists a curve of $n \times n$ invertible matrices $X(t)$ such that $\mathcal{B}(t) = \mathcal{A}(t)X(t)$. Differentiating this equation twice and using that $\mathcal{A}(t)$ is normal, we see that the only possible way in which the columns of $\ddot{\mathcal{B}}(t)$ could be linear combinations of the columns of $\mathcal{B}(t)$ is that $\dot{X}(t)$ be identically zero. □

The use of normal frames reduces to linear algebra the congruence problem for (parameterized) fanning curves: given two fanning curves $\ell(t)$ and $\tilde{\ell}(t)$ in G_n, when does there exist an invertible linear transformation $\mathbf{T}$ of $\mathbb{R}^{2n}$ such that $\tilde{\ell}(t) = \mathbf{T}\ell(t)$?

Theorem 4.3. *Two fanning curves of n-dimensional subspaces of $\mathbb{R}^{2n}$ are congruent if and only if the Schwarzians of any two of their normal frames are conjugate by a constant $n \times n$ invertible matrix.*

Proof. If $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are two normal frames spanning congruent fanning curves, then there exists a linear transformation $\mathbf{T}$ of $\mathbb{R}^{2n}$ such that $\mathbf{T}\mathcal{A}(t)$ and $\mathcal{B}(t)$ span the same curve. Since $\mathbf{T}\mathcal{A}(t)$ is again a normal frame, Corollary 3.6 together with Proposition 4.2 tell us that the Schwarzians of $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are conjugate by a constant $n \times n$ invertible matrix.

On the other hand, if $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are two normal frames such that $X^{-1}\{\mathcal{A}(t), t\}X = \{\mathcal{B}(t), t\}$, the normal frame $\mathcal{A}(t)X$ spans the same curve as $\mathcal{A}(t)$ and has the same Schwarzian as $\mathcal{B}(t)$. Without loss of generality we could have then assumed that $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are two normal frames with the same Schwarzian.

In order to show that these frames are congruent, we set $\mathbf{T} = (\mathcal{B}(0)|\dot{\mathcal{B}}(0))(\mathcal{A}(0)|\dot{\mathcal{A}}(0))^{-1}$ and remark that $\mathcal{D}(t) = \mathbf{T}\mathcal{A}(t)$ satisfies the same second-order differential equation, $\ddot{\mathcal{D}} + (1/2)\mathcal{D}\{\mathcal{B}(t), t\} = 0$, as $\mathcal{B}(t)$ and has the same initial conditions. It follows that $\mathcal{D}(t) = \mathcal{B}(t)$ and, therefore, $\mathcal{A}(t)$ is congruent to $\mathcal{B}(t)$. □

4.1. Fundamental theorem for unparameterized fanning curves

In order to state and prove the analogue of Theorem 4.3 for unparameterized fanning curves in G_n, we define a *special normal frame* as a normal frame whose Schwarzian has zero trace.

Proposition 4.4. *If $\ell(t)$ is a fanning curve of n-dimensional subspaces in $\mathbb{R}^{2n}$, there exists a special normal frame that spans a reparameterization of it. Moreover, if $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are special normal frames spanning the same unparameterized fanning curve in G_n, there exist a fixed invertible $n \times n$ matrix X and a projective transformation σ of the real line such that $\mathcal{B}(\sigma(t)) = \mathcal{A}(t)X(t)$.*

Proof. Let $\ell(t(s))$ be a special parameterization of the fanning curve traced by $\ell(t(s))$. The trace of the Jacobi endomorphism of $\ell(t(s))$ is then identically zero and, by Corollary 3.5, so is the trace of the Schwarzian of any frame spanning it. It follows that any normal frame spanning $\ell(t(s))$ is a special normal frame. The question of uniqueness is addressed in a way with which by now the reader is familiar. □

Theorem 4.5. *Two unparameterized fanning curves of n-dimensional subspaces of $\mathbb{R}^{2n}$ are congruent if and only if up to a projective change of parameters the Schwarzians of any two of their special normal frames are conjugate by a constant $n \times n$ invertible matrix.*

The proof is almost identical to that of Theorem 4.3.

5. Special classes of fanning curves

As a second application of the concept of normal frames we prove the following characterization of fanning curves with zero Jacobi endomorphism (compare with [6, Theorem 1]).

Theorem 5.1. *The following conditions on a fanning curve $\ell(t)$ of n-dimensional subspaces in $\mathbb{R}^{2n}$ are equivalent*:

(1) *The Jacobi endomorphism of $\ell(t)$ is zero.*
(2) *The Schwarzian of any curve of frames spanning $\ell(t)$ is zero.*
(3) *Any normal frame spanning $\ell(t)$ is a line in the space of frames.*
(4) *The horizontal curve of $\ell(t)$ is constant.*

Proof. The equivalence of (1) and (2) follows immediately from Theorem 3.4. That (2) implies (3) follows from Eq. (1.3). Indeed, if $\mathcal{A}(t)$ is a normal frame $\ddot{\mathcal{A}}(t) = -(1/2)\mathcal{A}(t)\{\mathcal{A}(t), t\}$. If the Schwarzian is zero, then $\mathcal{A}(t) = \mathcal{A}(0) + t\dot{\mathcal{A}}(0)$.

Since the horizontal derivative of a line $\mathcal{A}(t) = \mathcal{A}(0) + t\dot{\mathcal{A}}(0)$ is constantly equal to $\dot{\mathcal{A}}(0)$, (3) clearly implies (4). In order to see that (4) implies (2), thereby finishing the proof, notice that if the horizontal curve of $\ell(t)$ is constant, the horizontal derivative $\mathcal{H}(t)$ of any fanning frame $\mathcal{A}(t)$ spanning $\ell(t)$ is of the form $\mathcal{H}(t) = \mathcal{H}(0)X(t)$ for some curve $X(t)$ of $n \times n$ invertible matrices. In this case $\dot{\mathcal{H}}(t) = \mathcal{H}(0)\dot{X}(t) = \mathcal{H}(t)X(t)^{-1}\dot{X}(t)$ and, therefore, $\mathbf{P}(t)\dot{\mathcal{H}}(t) = 0$. Applying Eq. (3.1) and Theorem 3.4, we conclude that

$$0 = \mathbf{P}(t)\dot{\mathcal{H}}(t) = -\mathbf{K}(t)\mathcal{A}(t) = -(1/2)\mathcal{A}(t)\big\{\mathcal{A}(t), t\big\}.$$

The upshot is that the Schwarzian $\{\mathcal{A}(t), t\}$ must be zero. □

As a corollary, we obtain an easy proof of the following important property of the matrix Schwarzian [21, p. 205].

Corollary 5.2. *Let $M(t)$ be a smooth curve of $n \times n$ matrices such that $\dot{M}(t)$ is invertible for all values of t. The matrix Schwarzian of $M(t)$ is identically zero if and only if there exist matrices A, B, C, and D such that $M(t) = (C + tD)(A + tB)^{-1}$.*

Proof. Since the matrix Schwarzian of $M(t)$ is zero, so is the Schwarzian of the fanning frame

$$\mathcal{A}(t) = \begin{pmatrix} I \\ M(t) \end{pmatrix},$$

and of any other fanning frame spanning the same curve in the Grassmannian. In particular, the Schwarzian of a normal frame $\mathcal{B}(t) = \mathcal{A}(t)X(t)$ vanishes identically. By the preceding theorem, we have that $\mathcal{B}(t) = \mathcal{B}(0) + t\dot{\mathcal{B}}(0)$. If we write

$$\mathcal{B}(0) = \begin{pmatrix} A \\ C \end{pmatrix} \quad \text{and} \quad \dot{\mathcal{B}}(0) = \begin{pmatrix} B \\ D \end{pmatrix}, \quad \text{then} \quad \begin{pmatrix} I \\ M(t) \end{pmatrix} = \begin{pmatrix} A + tB \\ C + tD \end{pmatrix} X(t)^{-1}.$$

It follows that $X(t) = (A + tB)$ and $M(t) = (C + tD)(A + tB)^{-1}$. □

We now characterize the fanning curves of the form $\ell(t) = \exp(t\mathbf{X})\ell$. These curves are ubiquitous in the study of the Jacobi equation in symmetric and reductive spaces in Riemannian and Finsler geometry.

Definition 5.3. A fanning curve $\ell(t)$ in the Grassmannian G_n is said to be *parallel* if the Schwarzian of a normal frame spanning it is constant. It is called *weakly parallel* if whenever $\mathcal{A}(t)$ is a normal frame spanning $\ell(t)$, there exist $n \times n$ matrices X and Y such that $\{\mathcal{A}(t), t\} = \exp(-tY)X\exp(tY)$.

Lemma 5.4. *A fanning frame $\mathcal{A}(t)$ satisfies a constant-coefficient, second-order differential equation $\ddot{\mathcal{A}} + \dot{\mathcal{A}}P + \mathcal{A}Q = 0$ (P and Q are $n \times n$ matrices) if and only if it is of the form $\mathcal{A}(t) = \exp(t\mathbf{X})\mathcal{A}(0)$ for some linear transformation $\mathbf{X}$ from $\mathbb{R}^{2n}$ to itself such that the $2n \times 2n$ matrix $(\mathcal{A}(0)|\mathbf{X}\mathcal{A}(0))$ is invertible.*

Proof. Assume that $\mathcal{A}(t) = \exp(t\mathbf{X})\mathcal{A}(0)$ for some $\mathbf{X}$ satisfying the hypotheses of the theorem. Since the columns of $\mathcal{A}(0)$ and $\dot{\mathcal{A}}(0) = \mathbf{X}\mathcal{A}(0)$ span $\mathbb{R}^{2n}$, there exist $n \times n$ matrices P and Q so that $\mathbf{X}^2\mathcal{A}(0) + \mathbf{X}\mathcal{A}(0)P + \mathcal{A}(0)Q = 0$. We have then that

$$\ddot{\mathcal{A}}(t) + \dot{\mathcal{A}}(t)P + \mathcal{A}(t)Q = \exp(t\mathbf{X})\big(\mathbf{X}^2\mathcal{A}(0) + \mathbf{X}\mathcal{A}(0)P + \mathcal{A}(0)Q\big) = 0.$$

Conversely, we may write the equation $\ddot{\mathcal{A}}(t) + \dot{\mathcal{A}}(t)P + \mathcal{A}(t)Q = 0$ as

$$\frac{d}{dt}\big(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)\big) = \big(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)\big)\begin{pmatrix} 0 & -Q \\ I & -P \end{pmatrix}.$$

It follows that $\mathcal{A}(t) = \exp(t\mathbf{X})\mathcal{A}(0)$, where

$$\mathbf{X} = \big(\mathcal{A}(0)\big|\dot{\mathcal{A}}(0)\big)\begin{pmatrix} 0 & -Q \\ I & -P \end{pmatrix}\big(\mathcal{A}(0)\big|\dot{\mathcal{A}}(0)\big)^{-1}. \qquad \square$$

Theorem 5.5. *A fanning curve in* G_n *is weakly parallel if and only if it is of the form* $\ell(t) = \exp(t\mathbf{X})\ell(0)$*, where* $\mathbf{X}$ *is a linear transformation of* $\mathbb{R}^{2n}$ *such that* $\mathbf{X}\ell(0)$ *is transverse to* $\ell(0)$*.*

Proof. If the fanning curve $\ell(t)$ is weakly parallel and $\mathcal{A}(t)$ is a normal frame spanning it, there exist $n \times n$ matrices X and Y such that $\ddot{\mathcal{A}}(t) + (1/2)\mathcal{A}(t)\exp(-tY)X\exp(tY) = 0$. Setting $\mathcal{B}(t) = \mathcal{A}(t)\exp(-tY)$, we easily verify that $\mathcal{B}(t)$ satisfies the constant-coefficient, second-order differential equation

$$\ddot{\mathcal{B}}(t) + \dot{\mathcal{B}}(t)(2Y) + \mathcal{B}(t)\big(Y^2 + (1/2)X\big) = 0.$$

By Lemma 5.4, this implies that $\mathcal{B}(t) = \exp(t\mathbf{X})\mathcal{B}(0)$ and, therefore, $\ell(t) = \exp(t\mathbf{X})\ell(0)$ for some linear transformation $\mathbf{X}$ from $\mathbb{R}^{2n}$ to itself.

Conversely, assume that $\ell(t) = \exp(t\mathbf{X})\ell(0)$ and let $\mathcal{A}(0)$ be a $2n \times n$ matrix whose columns span the subspace $\ell(0)$. By Lemma 5.4, $\mathcal{A}(t) = \exp(t\mathbf{X})\mathcal{A}(0)$ satisfies a constant-coefficient, second-order differential equation $\ddot{\mathcal{A}} + \dot{\mathcal{A}}P + \mathcal{A}Q = 0$. Hence, the Schwarzian $\{\mathcal{A}(t), t\}$ is the constant matrix $X = 2Q - (1/2)P^2$, and the Schwarzian of the normal frame $\mathcal{A}(t)\exp(tP/2)$ is $\exp(-tP/2)X\exp(tP/2)$. Thus, the fanning curve $\ell(t)$ is weakly parallel. $\square$

Corollary 5.6. *If* $\mathcal{A}(t)$ *is any frame spanning a fanning curve* $\ell(t)$ *that is the projection of a one-parameter subgroup of* $\mathrm{GL}(2n)$ *to the Grassmannian* G_n*, then the Schwarzian of* $\mathcal{A}(t)$ *satisfies a Lax equation*

$$\frac{d}{dt}\big\{\mathcal{A}(t), t\big\} = \big[\big\{\mathcal{A}(t), t\big\}, Y(t)\big],$$

where $Y(t)$ *is a smooth curve of* $n \times n$ *matrices.*

Proof. Since $\ell(t)$ is weakly parallel, the Schwarzian of a normal frame $\mathcal{B}(t)$ spanning it is of the form $\exp(-tY)X\exp(tY)$ for fixed $n \times n$ matrices X and Y. Therefore, we have the equation

$$\frac{d}{dt}\big\{\mathcal{B}(t), t\big\} = \big[\big\{\mathcal{B}(t), t\big\}, Y\big].$$

If $\mathcal{A}(t) = \mathcal{B}(t)X(t)$ is any other fanning frame spanning $\ell(t)$,

$$\begin{aligned}
\frac{d}{dt}\big\{\mathcal{A}(t), t\big\} &= \frac{d}{dt}X(t)^{-1}\big\{\mathcal{B}(t), t\big\}X(t) \\
&= -X(t)^{-1}\dot{X}(t)\big\{\mathcal{A}(t), t\big\} + X(t)^{-1}\frac{d}{dt}\big\{\mathcal{B}(t), t\big\}X(t) + \big\{\mathcal{A}(t), t\big\}X(t)^{-1}\dot{X}(t) \\
&= -X(t)^{-1}\dot{X}(t)\big\{\mathcal{A}(t), t\big\} + X(t)^{-1}\big[\big\{\mathcal{B}(t), t\big\}, Y\big]X(t) + \big\{\mathcal{A}(t), t\big\}X(t)^{-1}\dot{X}(t) \\
&= \big[\big\{\mathcal{A}(t), t\big\}, X(t)^{-1}YX(t) + X(t)^{-1}\dot{X}(t)\big]. \qquad \square
\end{aligned}$$

Theorem 5.7. *A fanning curve in* G_n *is parallel if and only if it is of the form* $\ell(t) = \exp(t\mathbf{X})\ell(0)$*, where* $\mathbf{X}$ *is a linear transformation of* $\mathbb{R}^{2n}$ *such that* $\mathbf{X}\ell(0)$ *is transverse to* $\ell(0)$ *and* $\mathbf{X}^2\ell(0) \subset \ell(0)$*.*

Proof. If the fanning curve $\ell(t)$ is parallel, a normal frame $\mathcal{A}(t)$ spanning it satisfies a constant-coefficient, second-order differential equation $\ddot{\mathcal{A}} + \mathcal{A}Q = 0$. By Lemma 5.4, this implies that $\mathcal{A}(t) = \exp(t\mathbf{X})\mathcal{A}(0)$, where

$$\mathbf{X} = \left(\mathcal{A}(0)\middle|\dot{\mathcal{A}}(0)\right)\begin{pmatrix} 0 & -Q \\ I & 0 \end{pmatrix}\left(\mathcal{A}(0)\middle|\dot{\mathcal{A}}(0)\right)^{-1}.$$

Notice that $\mathbf{X}\mathcal{A}(0) = \dot{\mathcal{A}}(0)$ and $\mathbf{X}^2\mathcal{A}(0) = -\mathcal{A}(0)Q$. Therefore, $\mathbf{X}\ell(0)$ is transverse to $\ell(0)$ and $\mathbf{X}^2\ell(0) \subset \ell(0)$.

Conversely, let $\ell(t)$ be a fanning curve of the form $\exp(t\mathbf{X})\ell(0)$ for some linear transformation $\mathbf{X}$ from $\mathbb{R}^{2n}$ to itself. The condition that $\mathbf{X}^2\ell(0)$ be a subspace of $\ell(0)$ implies that if $\mathcal{A}(0)$ is any $2n \times n$ matrix whose columns span $\ell(0)$, then $\mathcal{A}(t) = \exp(t\mathbf{X})\mathcal{A}(0)$ is a normal frame. By Lemma 5.4, the Schwarzian of $\mathcal{A}(t)$ is constant and, therefore, $\ell(t)$ is parallel. □

Proposition 5.8. *The Jacobi endomorphism of a fanning curve $\ell(t)$ in G_n is zero if and only if it is of the form $\ell(t) = \exp(t\mathbf{X})\ell(0)$ with $\mathbf{X}$ a linear transformation $\mathbf{X}$ from $\mathbb{R}^{2n}$ to itself such that $\mathbf{X}\ell(0)$ is transversal to $\ell(0)$ and $\mathbf{X}^2$ is zero.*

Proof. If $\mathcal{A}(0)$ is a $2n \times n$ matrix of whose columns span ℓ, then $\mathcal{A}(t) = \exp(t\mathbf{X})\mathcal{A}(0)$ is a normal frame spanning $\ell(t)$ and satisfying $\ddot{\mathcal{A}}(t) = \mathbf{X}^2\mathcal{A}(t) = 0$. It follows that the Jacobi endomorphism of $\ell(t)$ is zero.

Conversely, if $\mathcal{A}(t)$ is a normal frame spanning a fanning curve with vanishing Jacobi endomorphism, $\mathcal{A}(t) = \mathcal{A}(0) + t\dot{\mathcal{A}}(0)$. If $\mathbf{X}$ is any linear transformation from $\mathbb{R}^{2n}$ to itself such that $\mathbf{X}\mathcal{A}(0) = \dot{\mathcal{A}}(0)$ and $\mathbf{X}^2 = 0$, then $\mathcal{A}(t) = \exp(t\mathbf{X})\mathcal{A}(0)$ and, therefore, $\ell(t) = \exp(t\mathbf{X})\ell(0)$. □

6. Fanning curves of Lagrangian subspaces

We now turn to the study of fanning curves of Lagrangian subspaces on $\mathbb{R}^{2n}$ provided with its standard symplectic structure

$$\omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T\mathbf{J}\mathbf{w}, \quad \text{where } \mathbf{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Recall that the set Λ_n of all Lagrangian subspaces of $\mathbb{R}^{2n}$—the *Lagrangian Grassmannian*—is a smooth submanifold of the Grassmannian G_n. If ℓ is a Lagrangian subspace of $\mathbb{R}^{2n}$ the tangent space $T_\ell\Lambda_n$ is canonically isomorphic to the space of symmetric bilinear forms on ℓ. It will be useful to describe this isomorphism in terms of frames.

Let us say that a *Lagrangian frame* is a $2n \times n$ matrix $\mathcal{A}$ of rank n such that the subspace spanned by the columns of $\mathcal{A}$ is Lagrangian. Equivalently, a $2n \times n$ matrix $\mathcal{A}$ is a Lagrangian frame if it has rank n and $\mathcal{A}^T\mathbf{J}\mathcal{A} = 0$. If $\mathcal{A}(t)$ is a smooth curve of Lagrangian frames spanning a curve $\ell(t)$ of Lagrangian subspaces, we have that

$$\mathcal{A}(t)^T\mathbf{J}\mathcal{A}(t) = 0 \quad \text{and} \quad \dot{\mathcal{A}}(t)^T\mathbf{J}\mathcal{A}(t) + \mathcal{A}(t)^T\mathbf{J}\dot{\mathcal{A}}(t) = 0. \tag{6.1}$$

The second equation is simply the derivative of the first.

Definition 6.1. The *Wronskian* of a smooth curve $\mathcal{A}(t)$ of Lagrangian frames is the curve of $n \times n$ matrices defined by $W(t) = -\mathcal{A}(t)^T\mathbf{J}\dot{\mathcal{A}}(t)$.

Note that in two dimensions the Wronskian of a curve of Lagrangian frames $(f_1(t), f_2(t))^T$ is the "usual" Wronskian $f_1(t)\dot{f}_2(t) - f_2(t)\dot{f}_1(t)$.

Proposition 6.2. *The Wronskian $W(t)$ of a smooth curve $\mathcal{A}(t)$ of Lagrangian frames satisfies the following properties*:

(1) *$W(t)$ is symmetric for all values of t.*
(2) *If $X(t)$ is a smooth curve of invertible $n \times n$ matrices, the Wronskian of the curve of Lagrangian frames $\mathcal{A}(t)X(t)$ is $X(t)^T W(t)X(t)$.*
(3) *If $\mathbf{S}$ is a linear symplectic transformation of $(\mathbb{R}^{2n}, \omega)$, the Wronskian of $\mathbf{S}\mathcal{A}(t)$ is also $W(t)$.*
(4) *The curve of Lagrangian frames $\mathcal{A}(t)$ is fanning if and only if $W(t)$ is invertible for all values of t.*
(5) *If s is a diffeomorphism of the real line, the Wronskian of $\mathcal{A}(s(t))$ is $W(s(t))\dot{s}(t)$.*

Proof. Properties (1) and (2) follow immediately from Eq. (6.1) and the definition of the Wronskian. Property (3) follows from the fact that $\mathbf{S}$ is symplectic (i.e., $\mathbf{S}^T\mathbf{J}\mathbf{S} = \mathbf{J}$), while property (5) is a trivial computation. Lastly, to show that $\mathcal{A}(t)$ is fanning if and only if $W(t)$ is never singular, it is enough to remark that $(\mathcal{A}(t)|\dot{\mathcal{A}}(t))$ is nonsingular if and only if

$$\big(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)\big)^T\mathbf{J}\big(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)\big) = \begin{pmatrix} 0 & -W(t) \\ W(t) & \dot{\mathcal{A}}(t)^T\mathbf{J}\dot{\mathcal{A}}(t) \end{pmatrix}$$

is nonsingular. □

The geometric interpretation of property (2) is that if $\ell(t)$ is a smooth curve of Lagrangian subspace of $\mathbb{R}^{2n}$ and $\mathcal{A}(t)$ is a curve of Lagrangian frames spanning it, the bilinear form on the subspace $\ell(\tau)$ defined by the matrix $W(\tau)$ in the basis formed by the columns of $\mathcal{A}(\tau)$ is well defined and does not depend on the curve of frames spanning $\ell(t)$. This gives the canonical isomorphism between $T_{\ell(\tau)}\Lambda_n$ and the space of bilinear forms on $\ell(\tau)$.

Properties (1) and (4) imply that the Wronskian $W(t)$ of a fanning curve of Lagrangian frames is a curve of invertible, symmetric matrices and, therefore, that the index of the matrices $W(t)$ is constant. By the remark in the preceding paragraph, the index of $W(t)$ depends only on the fanning curve on the Lagrangian Grassmannian and not on the particular fanning frame that spans it.

Definition 6.3. The *signature* of a fanning curve of Lagrangian subspaces is the index of the Wronskian of any curve of Lagrangian frames that spans it.

Before considering the congruence problem for fanning curves of Lagrangian subspaces, let us introduce a large class of examples that arise in applications and whose invariants are easily computed.

6.1. Systems of Lagrange equations

Let us consider a system of Lagrange equations of the form

$$\frac{d}{dt}\big(\dot{\mathbf{x}}^T K(t)\big) + \mathbf{x}^T V(t) = \mathbf{0} \tag{6.2}$$

where $K(t)$ and $V(t)$ are smooth curves of $n \times n$ symmetric matrices and $K(t)$ is invertible for all values of t. The vectors $\mathbf{x}^T$ and $\dot{\mathbf{x}}^T$ in $\mathbb{R}^n$ are written as $1 \times n$ matrices. We may also write Eq. (6.2) in Hamiltonian form

$$\dot{\mathbf{x}}^T = \mathbf{p}^T K(t)^{-1}, \qquad \dot{\mathbf{p}}^T = -\mathbf{x}^T V(t)$$

with Hamiltonian $H(\mathbf{x}, \mathbf{p}, t) = (1/2)(\mathbf{p}^T K(t)^{-1}\mathbf{p} + \mathbf{x}^T V(t)\mathbf{x})$.

Proposition 6.4. *If $\mathcal{A}(t)$ is a $2n \times n$ matrix whose rows are solutions of Eq.* (6.2) *and such that $(\mathcal{A}(0)|\dot{\mathcal{A}}(0)K(0))$ is a symplectic matrix, then $\mathcal{A}(t)$ is a fanning curve of Lagrangian frames whose Wronskian is $W(t) = K(t)^{-1}$ and whose Schwarzian is*

$$\{\mathcal{A}(t), t\} = 2V(t)K(t)^{-1} - (1/2)\big(\dot{K}(t)K(t)^{-1}\big)^2 - \frac{d}{dt}\big(\dot{K}(t)K(t)^{-1}\big).$$

Proof. Notice that Eq. (6.2) may be written as

$$\frac{d}{dt}\big(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)K(t)\big) = \big(\mathcal{A}(t)\big|\dot{\mathcal{A}}(t)K(t)\big)\begin{pmatrix} 0 & -V(t) \\ K(t)^{-1} & 0 \end{pmatrix}$$

and that, since both $K(t)$ and $V(t)$ are symmetric, the last matrix defines a curve in the Lie algebra of the group of symplectic transformations of $(\mathbb{R}^{2n}, \omega)$. It follows that if the initial condition $(\mathcal{A}(0)|\dot{\mathcal{A}}(0)K(0))$ is symplectic, then $(\mathcal{A}(t)|\dot{\mathcal{A}}(t)K(t))$ is symplectic for all values of t. This implies at once that $\mathcal{A}(t)$ is a curve of Lagrangian frames. Moreover, since $(\mathcal{A}(t)|\dot{\mathcal{A}}(t))$ is nonsingular if and only if $(\mathcal{A}(t)|\dot{\mathcal{A}}(t)K(t))$ is nonsingular, we have that $\mathcal{A}(t)$ is a fanning curve of frames.

Note that since $(\mathcal{A}(t)|\dot{\mathcal{A}}(t)K(t))$ is symplectic, $\mathcal{A}(t)^T\mathbf{J}\dot{\mathcal{A}}(t)K(t) = -I$ and, therefore, the Wronskian $W(t) = -\mathcal{A}(t)^T\mathbf{J}\dot{\mathcal{A}}(t) = K(t)^{-1}$. The Schwarzian of $\mathcal{A}(t)$ is easily computed using Eqs. (1.2) and (6.2). □

6.2. Lagrangian normal frames and solution of the congruence problem

Let us start with a simple—albeit useful—remark.

Lemma 6.5. *If $\mathcal{A}(t)$ is a curve of Lagrangian frames such the columns of $\ddot{\mathcal{A}}(t)$ are linear combinations of those of $\mathcal{A}(t)$, then $\dot{\mathcal{A}}(t)^T\mathbf{J}\dot{\mathcal{A}}(t) = 0$. In particular, the Wronskian of such a frame is constant.*

Proof. Differentiating the equation $\dot{\mathcal{A}}(t)^T\mathbf{J}\mathcal{A}(t) + \mathcal{A}(t)^T\mathbf{J}\dot{\mathcal{A}}(t) = 0$ and using that $\ddot{\mathcal{A}}(t)^T\mathbf{J}\mathcal{A}(t)$ and $\mathcal{A}(t)^T\mathbf{J}\ddot{\mathcal{A}}(t)$ are both zero, we conclude that

$$0 = \ddot{\mathcal{A}}(t)^T\mathbf{J}\mathcal{A}(t) + 2\dot{\mathcal{A}}(t)^T\mathbf{J}\dot{\mathcal{A}}(t) + \mathcal{A}(t)^T\mathbf{J}\ddot{\mathcal{A}}(t) = 2\dot{\mathcal{A}}(t)^T\mathbf{J}\dot{\mathcal{A}}(t).$$

Since the derivative of the Wronskian of $\mathcal{A}(t)$ is $-\dot{\mathcal{A}}(t)^T\mathbf{J}\dot{\mathcal{A}}(t) - \mathcal{A}(t)^T\mathbf{J}\ddot{\mathcal{A}}(t)$ and both terms are zero, the Wronskian is constant. □

In view of this lemma, if a Lagrangian frame $\mathcal{A}(t)$ is normal in the sense of Section 4, then its Wronskian is constant. This allows us to define a more restrictive class of normal frames in the Lagrangian setting.

Definition 6.6. A fanning Lagrangian frame $\mathcal{A}(t)$ is said to be a *Lagrangian normal frame* if the columns of $\ddot{\mathcal{A}}(t)$ are linear combinations of those of $\mathcal{A}(t)$ and its Wronskian is the diagonal $n \times n$ matrix $I_{n,k}$ whose first k $(0 \leqslant k \leqslant n)$ diagonal entries equal -1 and the remaining diagonal entries equal 1.

Proposition 6.7. *If $\ell(t)$ is a fanning curve of Lagrangian subspaces in $\mathbb{R}^{2n}$, there exists a Lagrangian normal frame that spans it. Moreover if the signature of $\ell(t)$ is k $(0 \leqslant k \leqslant n)$ and $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are two Lagrangian normal frames spanning $\ell(t)$, there exists a fixed invertible $n \times n$ matrix X that preserves the quadratic form defined by the matrix $I_{n,k}$ and such that $\mathcal{B}(t) = \mathcal{A}(t)X$.*

Proof. By Proposition 4.2, there exists a normal frame $\mathcal{A}(t)$ spanning $\ell(t)$ and, by Lemma 6.5, the Wronskian of $\mathcal{A}(t)$ is a fixed symmetric matrix W. If X is any $n \times n$ matrix such that $X^TWX = I_{n,k}$, where k is the index of W, then $\mathcal{A}(t)X$ is a Lagrangian normal frame spanning $\ell(t)$.

Two Lagrangian normal frames $\mathcal{A}(t)$ and $\mathcal{B}(t)$ spanning the same curve $\ell(t)$ on the Lagrangian Grassmannian are, in particular, two normal frames spanning the same curve of n-dimensional subspaces of $\mathbb{R}^{2n}$. It follows that there exists an invertible $n \times n$ matrix X such that $\mathcal{B}(t) = \mathcal{A}(t)X$. Since the Wronskians of $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are both equal to $I_{n,k}$, property (2) in Proposition 6.2 implies that $X^T I_{n,k} X = I_{n,k}$. □

Lagrangian normal frames spanning a given fanning curve of Lagrangian subspaces are therefore determined up to multiplication by a fixed matrix in the orthogonal group $O(n-k,k)$.

Theorem 6.8. *Two fanning curves of Lagrangian subspaces of $\mathbb{R}^{2n}$ are congruent if and only if they have the same signature k $(0 \leqslant k \leqslant n)$ and the Schwarzians of any two of their Lagrangian normal frames are conjugate by a constant $n \times n$ matrix in $O(n-k,k)$.*

Proof. The proof of this theorem is similar to that of Theorem 4.3, except in one important point: having reduced the proof to verifying that two Lagrangian normal frames $\mathcal{A}(t)$ and $\mathcal{B}(t)$ that have the same Wronskian and Schwarzian are congruent, it is somewhat more tricky to find a linear symplectic transformation such that $\mathbf{S}\mathcal{A}(0) = \mathcal{B}(0)$ and $\mathbf{S}\dot{\mathcal{A}}(0) = \dot{\mathcal{B}}(0)$. Just as in the proof of Theorem 4.3, this would imply that $\mathbf{S}\mathcal{A}(t)$ equals $\mathcal{B}(t)$—and prove the theorem—because both Lagrangian normal frames satisfy the same second-order differential equation and have the same initial conditions.

In order to find $\mathbf{S}$ we remark that the matrices $\mathbf{A} := (\mathcal{A}(0)|\dot{\mathcal{A}}(0)I_{n,k})$ and $\mathbf{B} := (\mathcal{B}(0)|\dot{\mathcal{B}}(0)I_{n,k})$ are symplectic. Indeed, using that $\mathcal{A}(0)^T\mathbf{J}\mathcal{A}(0) = 0$, $\dot{\mathcal{A}}(0)^T\mathbf{J}\dot{\mathcal{A}}(0) = 0$ (Lemma 6.5), and that both $-\mathcal{A}(0)^T\mathbf{J}\dot{\mathcal{A}}(0)$ and $\dot{\mathcal{A}}(0)^T\mathbf{J}\mathcal{A}(0)$ are equal to the Wronskian $I_{n,k}$, we have that

$$\big(\mathcal{A}(0)\big|\dot{\mathcal{A}}(0)I_{n,k}\big)^T\mathbf{J}\big(\mathcal{A}(0)\big|\dot{\mathcal{A}}(0)I_{n,k}\big) = \begin{pmatrix} 0 & -I_{n,k}^2 \\ I_{n,k}^2 & 0 \end{pmatrix} = \mathbf{J}.$$

The matrix $\mathbf{S} := \mathbf{B}\mathbf{A}^{-1}$ is the one we need. □

The use of normal Lagrangian frames also makes it easy to prove that the horizontal curve of a fanning curve of Lagrangian subspaces is itself a curve of Lagrangian subspaces, and that the Schwarzian of a fanning Lagrangian frame $\mathcal{A}(t)$ is symmetric with respect to the inverse of its Wronskian (i.e. that the matrix $\{\mathcal{A}(t), t\}W(t)^{-1}$ is symmetric).

Proposition 6.9. *The horizontal curve of a fanning curve of Lagrangian subspaces is itself a curve of Lagrangian subspaces. Equivalently, if $\mathcal{A}(t)$ is a fanning Lagrangian frame, then its horizontal derivative $\mathcal{H}(t)$ is a curve of Lagrangian frames.*

Proof. Recall that if $\mathcal{B}(t)$ is normal frame, then its horizontal derivative is simply $\dot{\mathcal{B}}(t)$. If $\mathcal{B}(t)$ is also a Lagrangian normal frame, then Lemma 6.5 implies that $\dot{\mathcal{B}}(t)$ is a curve of Lagrangian frames.

If $\mathcal{A}(t)$ is any curve of Lagrangian frames, there exist a normal Lagrangian frame $\mathcal{B}(t)$ and a curve of invertible $n \times n$ matrices so that $\mathcal{A}(t) = \mathcal{B}(t)X(t)$. By Proposition 2.9, the horizontal derivative of $\mathcal{A}(t)$ is $\dot{\mathcal{B}}(t)X(t)$, which is a curve of Lagrangian frames. □

Proposition 6.10. *If $W(t)$ denotes the Wronskian of a fanning Lagrangian frame $\mathcal{A}(t)$, the matrix $\{\mathcal{A}(t), t\}W(t)^{-1}$ is symmetric.*

Proof. Let us first assume that $\mathcal{B}(t)$ is a normal Lagrangian frame with Wronskian $I_{n,k}$. An argument identical to the last step in the proof of Theorem 6.8 shows that the matrix $(\mathcal{B}(t)|\dot{\mathcal{B}}(t)I_{n,k})$ is symplectic. Since $\mathcal{B}(t)$ is a normal frame, we have that $\ddot{\mathcal{B}} + (1/2)\mathcal{B}(t)\{\mathcal{B}(t), t\}$ and, therefore,

$$\frac{d}{dt}\big(\mathcal{B}(t)\big|\dot{\mathcal{B}}(t)I_{n,k}\big) = \big(\mathcal{B}(t)\big|\dot{\mathcal{B}}(t)I_{n,k}\big)\begin{pmatrix} 0 & -\{\mathcal{B}(t), t\}I_{n,k} \\ I_{n,k} & 0 \end{pmatrix}.$$

It follows that the rightmost matrix must be in the Lie algebra of the group of linear symplectic transformations and, consequently, that $\{\mathcal{B}(t), t\}I_{n,k}$ is symmetric.

If $\mathcal{A}(t)$ is any fanning Lagrangian frame, there exists a Lagrangian normal frame $\mathcal{B}(t)$ and a curve of $n \times n$ invertible matrices $X(t)$ such that $\mathcal{A}(t) = \mathcal{B}(t)X(t)$. It follows that the Wronskian of $\mathcal{A}(t)$ is $W(t) = X(t)^T I_{n,k} X(t)$ and that $\{\mathcal{A}(t), t\} = X^{-1}\{\mathcal{B}(t), t\}X(t)$. Therefore, the matrix $\{\mathcal{A}(t), t\}W(t)^{-1} = X^{-1}\{\mathcal{B}(t), t\}I_{n,k}(X^{-1})^T$ is symmetric. $\square$

7. Geometry of one- and two-jets of fanning curves

In order to understand the geometry of submanifolds in a homogeneous space, it is necessary to study the prolonged action of the group of symmetries on the spaces of jets. Except in simple classical cases, this is usually an arduous undertaking; specially in the case of noncompact symmetry groups. It is then a pleasant surprise that the action of $\mathrm{GL}(2n)$ on the spaces of jets of fanning curves in G_n can be easily described and that much geometric information and insight can be gained from its study.

7.1. One-jets of fanning curves and the fundamental endomorphism

The easiest way to describe the action of $\mathrm{GL}(2n)$ on the space $\mathrm{J}^k_f(\mathbb{R}; \mathrm{G}_n)$ of k-jets of fanning curves on the Grassmannian is to use the natural representation of $\mathrm{J}^k_f(\mathbb{R}; \mathrm{G}_n)$ as the quotient of the space $\mathrm{J}^k_f(\mathbb{R}; \mathcal{M}_{2n\times n})$ of k-jets of fanning frames by the action of the group $\mathrm{J}^k(\mathbb{R}; \mathrm{GL}(n))$ of k-jets of smooth curves of invertible $n \times n$ matrices. For instance, the space of one-jets of fanning frames consists of ordered pairs $(\mathcal{A}, \dot{\mathcal{A}})$ of $2n \times n$ matrices such that $(\mathcal{A}|\dot{\mathcal{A}})$ is invertible, and the group of one-jets of curves of invertible $n \times n$ matrices consists of ordered pairs $(X, \dot{X})$ of $n \times n$ matrices where X is invertible. The actions of $\mathrm{J}^1(\mathbb{R}; \mathrm{GL}(n))$ and $\mathrm{GL}(2n)$ on $\mathrm{J}^1_f(\mathbb{R}; \mathcal{M}_{2n\times n})$ are given by

$$(\mathcal{A}, \dot{\mathcal{A}}) \cdot (X, \dot{X}) = (\mathcal{A}X, \dot{\mathcal{A}}X + \mathcal{A}\dot{X}) \quad \text{and} \quad \mathbf{T} \cdot (\mathcal{A}, \dot{\mathcal{A}}) = (\mathbf{T}\mathcal{A}, \mathbf{T}\dot{\mathcal{A}}),$$

respectively. If we represent the elements of $\mathrm{J}^1_f(\mathbb{R}; \mathrm{G}_n) = \mathrm{J}^1_f(\mathbb{R}; \mathcal{M}_{2n\times n})/\mathrm{J}^1(\mathbb{R}; \mathrm{GL}(n))$ as equivalence classes $[(\mathcal{A}, \dot{\mathcal{A}})]$, the action of $\mathrm{GL}(2n)$ on the one-jets of fanning curves in the Grassmannian is simply described by $\mathbf{T} \cdot [(\mathcal{A}, \dot{\mathcal{A}})] = [(\mathbf{T}\mathcal{A}, \mathbf{T}\dot{\mathcal{A}})]$.

Proposition 7.1. *The group of invertible linear transformations of $\mathbb{R}^{2n}$ acts transitively on the space of one-jets of fanning frames and, a fortiori, on the space of one-jets of fanning curves in the Grassmannian G_n.*

Proof. If $(\mathcal{A}, \dot{\mathcal{A}}) \in \mathrm{J}^1_f(\mathbb{R}; \mathcal{M}_{2n\times n})$, then $\mathbf{T} := (\mathcal{A}|\dot{\mathcal{A}})$ is in $\mathrm{GL}(2n)$ and

$$\mathbf{T}^{-1} \cdot (\mathcal{A}, \dot{\mathcal{A}}) = \left(\begin{pmatrix} I \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I \end{pmatrix} \right). \qquad \square$$

Theorem 7.2. *A map from the space of one-jets of fanning curves in G_n to the Lie algebra $\mathcal{G}l(2n)$ is equivariant with respect to the $\mathrm{GL}(2n)$ action on these spaces if and only if it is of the form $a\mathbf{I} + b\mathbf{F}$, where $\mathbf{I}$ is the identity matrix and a and b are real numbers.*

Proof. Let $\mathbf{G} : \mathrm{J}^1_f(\mathbb{R}; \mathcal{M}_{2n\times n}) \to \mathcal{G}l(2n)$ be a map that is invariant under the action of $\mathrm{J}^1(\mathbb{R}; \mathrm{GL}(n))$ and equivariant with respect to the action of $\mathrm{GL}(2n)$. Writing $\mathbf{G}(\mathcal{A}, \dot{\mathcal{A}})$ as

$$\mathbf{G}(\mathcal{A}, \dot{\mathcal{A}}) = (\mathcal{A}|\dot{\mathcal{A}}) \begin{pmatrix} G_{11}(\mathcal{A}, \dot{\mathcal{A}}) & G_{12}(\mathcal{A}, \dot{\mathcal{A}}) \\ G_{21}(\mathcal{A}, \dot{\mathcal{A}}) & G_{22}(\mathcal{A}, \dot{\mathcal{A}}) \end{pmatrix} (\mathcal{A}|\dot{\mathcal{A}})^{-1}$$

and using that $\mathbf{G}(\mathbf{T}\mathcal{A}, \mathbf{T}\dot{\mathcal{A}}) = \mathbf{T}\mathbf{G}(\mathcal{A}, \dot{\mathcal{A}})\mathbf{T}^{-1}$, we obtain that

$$\begin{pmatrix} G_{11}(\mathbf{T}\mathcal{A}, \mathbf{T}\dot{\mathcal{A}}) & G_{12}(\mathbf{T}\mathcal{A}, \mathbf{T}\dot{\mathcal{A}}) \\ G_{21}(\mathbf{T}\mathcal{A}, \mathbf{T}\dot{\mathcal{A}}) & G_{22}(\mathbf{T}\mathcal{A}, \mathbf{T}\dot{\mathcal{A}}) \end{pmatrix} = \begin{pmatrix} G_{11}(\mathcal{A}, \dot{\mathcal{A}}) & G_{12}(\mathcal{A}, \dot{\mathcal{A}}) \\ G_{21}(\mathcal{A}, \dot{\mathcal{A}}) & G_{22}(\mathcal{A}, \dot{\mathcal{A}}) \end{pmatrix}$$

for all matrices $\mathbf{T} \in \mathrm{GL}(2n)$. Since $\mathrm{GL}(2n)$ acts transitively on the space of one-jets of fanning frames, this implies that the blocks G_{ij} are constant $n \times n$ matrices. Moreover, the invariance under the action of $\mathrm{J}^1(\mathbb{R}; \mathrm{GL}(n))$ imposes the condition

$$\begin{pmatrix} X & \dot{X} \\ 0 & X \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} X & \dot{X} \\ 0 & X \end{pmatrix}^{-1} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

for all choices of matrices X and $\dot{X}$ with X invertible. This easily implies that

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} aI & bI \\ 0 & aI \end{pmatrix}$$

for some real numbers a and b and, therefore, that $\mathbf{G} = a\mathbf{I} + b\mathbf{F}$. $\square$

7.2. Two-jets of fanning curves and the horizontal map

The space of two-jets of fanning frames is the set of ordered triples $(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})$ of $2n \times n$ matrices such that $(\mathcal{A}|\dot{\mathcal{A}})$ is invertible. Likewise, the two-jets of curves of $n \times n$ invertible matrices are ordered triples of $n \times n$ matrices $(X, \dot{X}, \ddot{X})$ such that X is invertible. The actions of $\mathrm{J}^2(\mathbb{R}; \mathrm{GL}(n))$ and $\mathrm{GL}(2n)$ on $\mathrm{J}^2_f(\mathbb{R}; \mathcal{M}_{2n\times n})$ are given by

$$(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}}) \cdot (X, \dot{X}, \ddot{X}) = (\mathcal{A}X, \dot{\mathcal{A}}X + \mathcal{A}\dot{X}, \ddot{\mathcal{A}}X + 2\dot{\mathcal{A}}\dot{X} + \mathcal{A}\ddot{X}),$$
$$\mathbf{T} \cdot (\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}}) = (\mathbf{T}\mathcal{A}, \mathbf{T}\dot{\mathcal{A}}, \mathbf{T}\ddot{\mathcal{A}}),$$

respectively. However, unlike the case of one-jets, the action of $\mathrm{GL}(2n)$ on the space of two-jets of fanning frames is not transitive. Indeed, it is easy to show that the two-jets $(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})$ and $(\mathcal{B}, \dot{\mathcal{B}}, \ddot{\mathcal{B}})$ are in the same $\mathrm{GL}(2n)$-orbit if and only if the matrices $(\mathcal{A}|\dot{\mathcal{A}})^{-1}\ddot{\mathcal{A}}$ and $(\mathcal{B}|\dot{\mathcal{B}})^{-1}\ddot{\mathcal{B}}$ are equal. Nevertheless, we still have that $\mathrm{GL}(2n)$ acts transitively on $\mathrm{J}^2_f(\mathbb{R}; \mathrm{G}_n)$.

Proposition 7.3. *The group of invertible linear transformations of $\mathbb{R}^{2n}$ acts transitively on the space of two-jets of fanning curves in G_n.*

Proof. Since $\mathrm{J}^2_f(\mathbb{R}; \mathrm{G}_n) = \mathrm{J}^2_f(\mathbb{R}; \mathcal{M}_{2n\times n})/\mathrm{J}^2(\mathbb{R}; \mathrm{GL}(n))$, all that needs to be shown is that the joint action of $\mathrm{GL}(2n)$ and $\mathrm{J}^2(\mathbb{R}; \mathrm{GL}(n))$ on the space of two-jets of fanning frames is transitive. To verify this, just note that if $(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}}) \in \mathrm{J}^2_f(\mathbb{R}; \mathcal{M}_{2n\times n})$ and $\ddot{\mathcal{A}} + \dot{\mathcal{A}}P + \mathcal{A}Q = 0$, then

$$\big(\mathcal{A}|\dot{\mathcal{A}} + (1/2)\mathcal{A}P\big)^{-1} \cdot (\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}}) \cdot \big(I, (1/2)P, Q\big) = \left(\begin{pmatrix} I \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right). \qquad \square$$

We are now ready to prove the characterization of the horizontal curve of a fanning curve in the Grassmannian G_n that we stated in the introduction.

Theorem 7.4. *The assignment that sends a fanning curve $\ell(t)$ to its horizontal curve $h(t)$ is characterized by the following four properties*:

(1) *At each time t the subspace $h(t)$ is transversal to $\ell(t)$.*
(2) *The subspace $h(\tau)$ depends only on the two-jet of the curve $\ell(t)$ at $t = \tau$.*

(3) *If* $\mathbf{T}$ *is an invertible linear transformation of* $\mathbb{R}^{2n}$*, the horizontal curve of* $\mathbf{T}\ell(t)$ *is* $\mathbf{T}h(t)$*.*
(4) *If* $\ell(t)$ *is spanned by a line* $\mathcal{A} + t\mathcal{B}$ *in the space of frames,* $h(t)$ *is constant.*

Lemma 7.5. *A* $\mathrm{GL}(2n)$*-equivariant map* $j : \mathrm{J}^2_f(\mathbb{R}; \mathrm{G}_n) \to \mathrm{G}_n$ *is such that the subspaces* $j([(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})])$ *and* $[\mathcal{A}]$ *are always transversal if and only if it is of the form*

$$\big[(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})\big] \longmapsto \big[\dot{\mathcal{A}} + (1/2)\mathcal{A}P + a\mathcal{A}\big], \tag{7.1}$$

where a *is some real number and* P *is the* $n \times n$ *matrix defined by the equation* $\ddot{\mathcal{A}} + \dot{\mathcal{A}}P + \mathcal{A}Q = 0$*.*

Proof. Notice that, at the level of two-jets of fanning frames, $(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}}) \mapsto \dot{\mathcal{A}} + (1/2)\mathcal{A}P$ is just the horizontal derivative we studied in Section 2. For any real number a, the subspace $[\dot{\mathcal{A}} + (1/2)\mathcal{A}P + a\mathcal{A}]$ is transversal to $[\mathcal{A}]$ and, using the properties of the horizontal derivative, it is easy to show that the map defined by Eq. (7.1) is $\mathrm{GL}(2n)$ equivariant.

In order to prove the converse, let us define $\mathbf{P} : \mathrm{J}^2_f(\mathbb{R}; \mathcal{M}_{2n\times n}) \to \mathcal{G}l(2n)$ as the map whose value at a two-jet $(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}}) \in \mathrm{J}^2_f(\mathbb{R}; \mathcal{M}_{2n\times n})$ is the projection with range $[\mathcal{A}]$ and kernel $j([(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})])$. The properties of j are equivalent to the following properties of $\mathbf{P}$:

(i) $\mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})^2 = \mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})$,
(ii) $\mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})\mathcal{A} = \mathcal{A}$,
(iii) $\mathbf{P}(\mathbf{T}\mathcal{A}, \mathbf{T}\dot{\mathcal{A}}, \mathbf{T}\ddot{\mathcal{A}}) = \mathbf{T}\mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})\mathbf{T}^{-1}$,
(iv) $\mathbf{P}(\mathcal{A}X, \dot{\mathcal{A}}X + \mathcal{A}\dot{X}, \ddot{\mathcal{A}}X + 2\dot{\mathcal{A}}\dot{X} + \mathcal{A}\ddot{X}) = \mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})$.

Using (i) and (ii), we see that there exists a function $R = R(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})$ with values in the space of $n \times n$ matrices such that

$$\mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}}) = (\mathcal{A}|\dot{\mathcal{A}}) \begin{pmatrix} I & R \\ 0 & 0 \end{pmatrix} (\mathcal{A}|\dot{\mathcal{A}})^{-1}.$$

Moreover, since $(\mathcal{A}|\dot{\mathcal{A}})^{-1}\ddot{\mathcal{A}}$ is the complete invariant for the action of $\mathrm{GL}(2n)$ on two-jets of fanning frames, property (iii) implies that $R(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})$ depends only on $(\mathcal{A}|\dot{\mathcal{A}})^{-1}\ddot{\mathcal{A}}$. With a slight abuse of notation, we write $R = R((\mathcal{A}|\dot{\mathcal{A}})^{-1}\ddot{\mathcal{A}})$.

We now claim that (iv) is equivalent to the condition that $R((\mathcal{A}|\dot{\mathcal{A}})^{-1}\ddot{\mathcal{A}})$ is of the form $-(1/2)P - aI$, where a is an arbitrary real number. This is a somewhat tedious verification whose only underlying idea is to break up property (iv) into three simpler invariance properties: whenever X, $\dot{X}$, and $\ddot{X}$ are $n \times n$ matrices with X invertible,

(a) $\mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}} + \mathcal{A}\ddot{X}) = \mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})$,
(b) $\mathbf{P}(\mathcal{A}X, \dot{\mathcal{A}}X, \ddot{\mathcal{A}}X) = \mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})$,
(c) $\mathbf{P}(\mathcal{A}, \dot{\mathcal{A}} + \mathcal{A}\dot{X}, \ddot{\mathcal{A}} + 2\dot{\mathcal{A}}\dot{X}) = \mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})$.

From (a) we obtain that $R((\mathcal{A}|\dot{\mathcal{A}})^{-1}\ddot{\mathcal{A}})$ is really just a function of the last n rows of its argument. In other words, with a some abuse of notation, we may write $R((\mathcal{A}|\dot{\mathcal{A}})^{-1}\ddot{\mathcal{A}}) = R(-P)$. From (b) we obtain that $R(-X^{-1}PX) = X^{-1}R(-P)X$ for all $n \times n$ invertible matrices X. In particular, this implies that $R(0) = -aI$ for some real number a. From (c) we obtain that $R(-P + 2\dot{X}) = R(-P) + \dot{X}$. On setting $2\dot{X} = P$, we have that $R(-P) = -(1/2)P + R(0) = -(1/2)P - aI$.

Since $j([(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}})])$ is the kernel of

$$\mathbf{P}(\mathcal{A}, \dot{\mathcal{A}}, \ddot{\mathcal{A}}) = (\mathcal{A}|\dot{\mathcal{A}}) \begin{pmatrix} I & -(1/2)P - aI \\ 0 & 0 \end{pmatrix} (\mathcal{A}|\dot{\mathcal{A}})^{-1},$$

it must equal $[\dot{\mathcal{A}} + (1/2)\mathcal{A}P + a\mathcal{A}]$. $\square$

Proof of Theorem 7.4. Using properties (1)–(3) of $h(t)$, it follows from the previous lemma that $h(t) = [\dot{\mathcal{A}}(t) + (1/2)\mathcal{A}(t)P(t) + a\mathcal{A}(t)]$ for any choice of frame $\mathcal{A}(t)$ spanning $\ell(t)$. When $\mathcal{A}(t)$ is a line $\mathcal{A} + t\mathcal{B}$ in the space of frames, then $h(t) = [\mathcal{B} + a(\mathcal{A} + t\mathcal{B})]$ is constant if and only if a is zero. □

8. A short survey of other approaches

In this final section we briefly consider other approaches to the geometry of fanning curves and exhibit their relationship to the invariants studied in this paper. The reader is referred to the original papers for many of the proofs and technical details.

8.1. S. Ahdout's definition of the horizontal curve

In the interesting paper [8], S. Ahdout studies fanning curves of Lagrangian subspaces and submanifolds and, as an application, shows that a convex body in Euclidean n-space is determined up to rigid motion by the billiard system it defines. Most of the paper is concerned with the study of fanning curves of Lagrangian submanifolds through the use of normal forms, but it is there that fanning curves of Lagrangian subspaces and their horizontal curves explicitly appear in the literature for the first time.

To a triple of n-dimensional subspaces $(\ell_0, \ell; \ell_\infty)$ in $\mathbb{R}^{2n}$ such that ℓ_∞ is transverse to both ℓ_0 and ℓ, we assign the linear transformation $\phi(\ell_0, \ell; \ell_\infty) : \ell_0 \to \mathbb{R}^{2n}/\ell_0$ that equals the composition of canonical projection from $\mathbb{R}^{2n}$ to the quotient space $\mathbb{R}^{2n}/\ell_0$ and the unique linear transformation from ℓ_0 to ℓ_∞ whose graph in $\ell_0 \oplus \ell_\infty = \mathbb{R}^{2n}$ equals ℓ. In what follows ϕ is considered as a function of the variable ℓ whose domain is the open subset of G_n consisting of all n-dimensional subspaces transversal to ℓ_0. This last subspace and ℓ_∞ are treated as a parameters. It is well known that if ℓ'_∞ is transversal to both ℓ_0 and ℓ, the differentials of $\phi(\ell_0, \cdot; \ell_\infty)$ and $\phi(\ell_0, \cdot; \ell'_\infty)$ at ℓ are equal and describe the canonical isomorphism between $T_{\ell_0}\mathrm{G}_n$ and $\hom \ell_0 \mathbb{R}^{2n}/\ell_0$.

A curve $\ell(t)$ in G_n is fanning if at every time τ the linear map $\frac{d}{dt}\phi(\ell(\tau), \ell(t); \ell_\infty)$ is invertible at $t = \tau$. Here ℓ_∞ is any n-dimensional subspace transversal to $\ell(\tau)$ and, therefore, transversal to $\ell(t)$ for t close to τ. In Theorem (1.3) of [8], Ahdout proves that *if $\ell(t)$ is a fanning curve in G_n, at every fixed time τ there exists a unique n-dimensional subspace $h(\tau)$ that is transversal to $\ell(\tau)$ and such that*

$$\frac{d^2}{dt^2}\phi\big(\ell(\tau), \ell(t); h(\tau)\big)$$

is zero at $t = \tau$. A simple application of Theorem 7.4 shows that this is indeed another definition of the horizontal curve.

8.2. The approach of A.A. Agrachev et al. to the geometry of fanning curves

In the series of papers [1,2,4–6], A.A. Agrachev, R. Gamkrelidze, and I. Zelenko study the geometry of curves in the Lagrangian Grassmannian by means of an ingenious approach based on the use of Laurent series of matrices. The following interpretation of their approach to the study of fanning curves touches only a fraction of their work, most of which is concerned with special classes of non-fanning curves arising from applications in control theory. We mention in passing that Agrachev uses the term "regular Jacobi curves" to refer to fanning curves in the Lagrangian Grassmannian and the term "derivative curve" to refer to the horizontal curve.

Let $(\ell_0; \ell_\infty, \ell)$ be a triple of n-dimensional subspaces in $\mathbb{R}^{2n}$ such that ℓ_∞ and ℓ are transverse to ℓ_0, and let A be the linear transformation from ℓ_∞ to ℓ_0 whose graph in $\ell_\infty \oplus \ell_0 = \mathbb{R}^{2n}$ is ℓ. The nilpotent linear transformation $\mathbf{N}(\ell_0; \ell_\infty, \ell)$ from $\mathbb{R}^{2n}$ to itself defined by the fact that its restriction to ℓ_0 is identically zero and its restriction to ℓ_∞ coincides with A is characterized by three properties: (1) its kernel contains ℓ_0, (2) its range is contained in ℓ_0, and (3) the image of ℓ_∞ under the transformation $\mathbf{I} + \mathbf{N}(\ell_0; \ell_\infty, \ell)$ is equal to ℓ. Central to Agrachev's approach are the two following properties of $\mathbf{N}$:

Proposition 8.1. *If ℓ_∞, ℓ'_∞ and ℓ are three n-dimensional subspaces of $\mathbb{R}^{2n}$ transversal to a fourth n-dimensional subspace ℓ_0, and $\mathbf{T}$ is a linear transformation from $\mathbb{R}^{2n}$ to itself, then $\mathbf{N}(\mathbf{T}\ell_0; \mathbf{T}\ell_\infty, \mathbf{T}\ell) = \mathbf{T}\mathbf{N}(\ell_0; \ell_\infty, \ell)\mathbf{T}^{-1}$ and $\mathbf{N}(\ell_0; \ell'_\infty, \ell) = \mathbf{N}(\ell_0; \ell_\infty, \ell) + \mathbf{N}(\ell_0; \ell'_\infty, \ell_\infty)$.*

If $\ell(t)$ is a fanning curve and τ is a real number the curve of nilpotent operators $\mathbf{N}(\ell(\tau); \ell_\infty, \ell(t))$ has a simple pole at $t = \tau$ and its residue $\mathbf{N}_{-1}(\tau)$ sends ℓ_∞ isomorphically onto $\ell(\tau)$. In fact, for values of t close to τ, $\mathbf{N}$ can be *formally written* as the Laurent series

$$\mathbf{N}\big(\ell(\tau); \ell_\infty, \ell(t)\big) = \sum_{k=-1}^{\infty} (t-\tau)^k \mathbf{N}_k(\tau),$$

where the coefficients $\mathbf{N}_k(\tau)$ are nilpotent linear transformations from $\mathbb{R}^{2n}$ to itself whose kernels contain ℓ_0 and whose ranges are contained in ℓ_0. When $\ell(t)$ is spanned by a line $\mathcal{A}_1 + t\mathcal{A}_2$ in the space of frames,

$$\mathbf{N}\big(\ell(\tau); \ell_\infty, \ell(t)\big) = \frac{1}{t-\tau}\mathbf{N}_{-1}(\tau) + \mathbf{N}_0(\tau).$$

The notation $\mathbf{N}_k(\tau)$ is somewhat inaccurate in that $\mathbf{N}_k(\tau)$ depends on the $(k+2)$-jet of the curve $\ell(t)$ at $t = \tau$ and, at least a priori, on the choice of ℓ_∞. However, we have the following simple result:

Proposition 8.2. *With the previous notation, if $k \neq 0$, the transformation $\mathbf{N}_k(\tau)$ does not depend on the choice of subspace ℓ_∞ transversal to $\ell(\tau)$. On the other hand, whereas $\mathbf{N}_0(\tau)$ depends on this choice, the subspace $(\mathbf{I} + \mathbf{N}_0(\tau))\ell_\infty$ does not.*

Proof. By Proposition 8.1, the Laurent series of $\mathbf{N}(\ell(\tau); \ell'_\infty, \ell(t))$ centered at $t = \tau$ differs from that of $\mathbf{N}(\ell(\tau); \ell'_\infty, \ell(t))$ only in its zeroth term $\mathbf{N}'_0(\tau) = \mathbf{N}_0(\tau) + \mathbf{N}(\ell_0; \ell'_\infty, \ell_\infty)$. Writing

$$\mathbf{I} + \mathbf{N}'_0(\tau) = \mathbf{I} + \mathbf{N}_0(\tau) + \mathbf{N}(\ell_0; \ell'_\infty, \ell_\infty) \quad \text{as } \big(\mathbf{I} + \mathbf{N}_0(\tau)\big)\big(\mathbf{I} + \mathbf{N}(\ell_0; \ell'_\infty, \ell_\infty)\big),$$

we see that $(\mathbf{I} + \mathbf{N}'_0(\tau))\ell'_\infty = (\mathbf{I} + \mathbf{N}_0(\tau))\ell_\infty$. $\square$

Each one of the transformations $\mathbf{N}_k(\tau)$, $k \neq 0$, is an invariant of the fanning curve $\ell(t)$ in the sense that they only depend on its $(k+2)$-jet at $t = \tau$, and if we change $\ell(t)$ for $\mathbf{T}\ell(t)$, we change $\mathbf{N}_k$ for $\mathbf{T}\mathbf{N}_k(\tau)\mathbf{T}^{-1}$. Using the axiomatic characterization of the fundamental endomorphism and the horizontal curve given in Theorems 7.2 and 7.4, it is easy to uncover the relationship between these invariants and those considered in this paper.

Proposition 8.3. *Let $\ell(t)$ be a fanning curve in G_n. If ℓ_∞ is any n-dimensional subspace transversal to $\ell(\tau)$, the nilpotent transformation $\mathbf{N}_{-1}(\tau)$ coincides with the fundamental endomorphism $\mathbf{F}(\tau)$, and the subspace $(\mathbf{I} + \mathbf{N}_0(\tau))\ell_\infty$ is the horizontal subspace to $\ell(t)$ at $t = \tau$.*

8.3. Cartan's method of moving frames

While we have not found any reference that employs Cartan's method of moving frames to study fanning curves in G_n $(n > 1)$, it is straightforward to generalize the approach of Flanders in [10] to higher dimensions.

If $\ell(t)$ is a fanning curve in G_n, we look for what Flanders calls a *natural moving frame*. This is a pair of curves of frames $\mathcal{A}(t)$ and $\mathcal{B}(t)$ such that (1) $\mathcal{A}(t)$ spans $\ell(t)$, (2) $\dot{\mathcal{A}}(t) = \mathcal{B}(t)$, and (3) $\dot{\mathcal{B}}(t) = \mathcal{A}(t)R(t)$ for some curve of $n \times n$ matrices $R(t)$. This is clearly equivalent to finding what we have called a normal frame for $\ell(t)$. Therefore, we already know that natural moving frames always exist and can be found at the cost of solving a linear differential equation.

The curve of $2n \times 2n$ matrices $\mathbf{A}(t) = (\mathcal{A}(t)|\dot{\mathcal{A}}(t))$ is what is usually called the moving frame for $\ell(t)$. By Proposition 4.2, if $\mathbf{B}(t) = (\mathcal{B}(t)|\dot{\mathcal{B}}(t))$ is another moving frame for $\ell(t)$, there exists a fixed $n \times n$ invertible matrix X such that

$$\mathbf{B}(t) = \mathbf{A}(t)\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}.$$

Pulling back the Maurer–Cartan form on $\mathrm{GL}(2n)$ via the map $\mathbf{A}(t)$, we obtain

$$\mathbf{A}(t)^{-1}\dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & R(t) \\ I & 0 \end{pmatrix}.$$

Hence the "invariant" obtained Cartan's method is simply $R(t) = -(1/2)\{\mathcal{A}(t), t\}$. However, it must be taken into account that if we had chosen the moving frame $\mathbf{B}(t)$ we would have obtained $X^{-1}R(t)X$ instead. Note that the last n columns of any moving frame for $\ell(t)$ span the horizontal curve. Thus, the definition of the horizontal curve is implicit in the moving-frame approach to the geometry of fanning curves.

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